FOR THE
IB DIPLOMA

Mathematics

APPLICATIONS AND INTERPRETATION HL

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Contents

Introduction

Welcome to your coursebook for Mathematics for the IB Diploma: applications and interpretation HL. The structure and content of this coursebook follow the structure and content of the 2019 IB Mathematics: analysis and approaches guide, with headings that correspond directly with the content areas listed therein.

This is the second book required by students taking the higher level course. Students should be familiar with the content of Mathematics for the IB Diploma: applications and interpretation SL before moving on to this book.

Using this book

Special features of the chapters include:

ESSENTIAL UNDERSTANDINGS

Each chapter begins with a summary of the key ideas to be explored and a list of the knowledge and skills you will learn. These are revisited in a checklist at the end of each chapter.

CONCEPTS

The IB guide identifies 12 concepts central to the study of mathematics that will help you make connections between topics, as well as with the other subjects you are studying. These are highlighted and illustrated with examples at relevant points throughout the book. The concepts are: Approximation, Change, Equivalence, Generalization,Modelling, Patterns, Relationships, Space, Systems and Validity.

KEY POINTS

Important mathematical rules and formulae are presented as Key Points, making them easy to locate and refer back to when necessary.

WORKED EXAMPLES

There are many Worked Examples in each chapter, demonstrating how the Key Points and mathematical content described can be put into practice. Each Worked Example comprises two columns:

On the left, how to **think** about the problem and what tools or methods will be needed at each step	On the right, what to **write**, prompted by the left column, to produce a formal solution to the question.

Exercises

Each section of each chapter concludes with a comprehensive exercise so that you can test your knowledge of the content described and practise the skills demonstrated in the Worked Examples. Each exercise contains the following types of questions:

- **Drill questions:** These are clearly linked to particular Worked Examples and gradually increase in difficulty. Each of them has two parts – **a** and **b** – desgined such that if you get **a** wrong, **b** is an opportunity to have another go at a very similar question. If you get **a** right, there is not need to do **b** as well.
- **Problem-solving questions:** These questions require you to apply the skills you have mastered in the drill questions to more-complex, exam-style questions. They are colour-coded for difficulty.

1 Green questions are closely related to standard techniques and require a small number of processes. They should be approachable for all candidates.

2 Blue questions require students to make a small number of tactical decisions about how to apply the standard methods and they will often require multiple procedures. Candidates targeting the medium HL grades should find these questions challenging but achievable.

3 Red questions often require a creative problem solving approach and extended, technical procedures. Candidates targeting the top HL grades should find these questions challenging.

4 Black questions go beyond what is expected in IB examinations, but provide an enrichment opportunity for the very best students.

The questions in the Mixed Practice section at the end of each chapter are similarly colour-coded, and contain questions taken directly from past IB Diploma Mathematics exam papers. There are also two review exercises, one halfway through the book and one at the end, testing the skills and knowledge you have developed in the preceding chapters.

Answers to all exercises can be found at the back of the book.

A calculator symbol is used where we want to remind you that there is a particularly important calculator trick required in the question.

A non-calculator icon suggests a question is testing a particular skill that you should be able to do without the use of a calculator.

The guide places great emphasis on the importance of technology in mathematics and expects you to have a high level of fluency with the use of your calculator and other relevant forms of hardware and software. Therefore, we have included plenty of screenshots and questions aimed at raising awareness and developing confidence in these skills, within the contexts in which they are likely to occur. This icon is used to indicate topics for which technology is particularly useful or necessary.

Making connections: Mathematics is all about making links. You might be interested to see how something you have just learned will be used elsewhere in the course and in different topics, or you may need to go back and remind yourself of a previous topic.

Be the Examiner

These are activities that present you with three different worked solutions to a particular question or problem. Your task is to determine which one is correct and to work out where the other two went wrong.

Proof

Proofs are set out in a similar way to Worked Examples, helping you to gain a deeper understanding of the mathematical rules and statements you will be using and to develop the thought processes required to write your own proofs.

TOOLKIT

There are questions, investigations and activities interspersed throughout the chapters to help you develop mathematical thinking skills, building on the introductory Toolkit chapter from the Mathematics for the IB Diploma: analysis and approaches SL book in relevant contexts. Although the ideas and skills presented will not be examined, these features are designed to give you a deeper insight into the topics that will be. Each Toolkit box addresses one the following three key topics: proof, modelling and problem-solving.

International mindedness

These boxes explore how the exchange of information and ideas across national boundaries has been essential to the progress of mathematics and to illustrate the international aspects of the subject.

You are the Researcher

This feature prompts you to carry out further research into subjects related to the syllabus content. You might like to use some of these ideas as starting points for your mathematical exploration or even an extended essay.

LEARNER PROFILE

Opportunities to think about how you are demonstrating the attributes of the IB Learner Profile are highlighted at the beginning of appropriate chapters.

Tips

There are short hints and tips provided in the margins throughout the book.

TOK Links

Links to the interdisciplinary Theory of Knowledge element of the IB Diploma course are throughout the book.

Links to: Other subjects

Links to other IB Diploma subjects are made at relevant points, highlighting some of the real-life applications of the mathematical skills you will learn.

Topics that have direct real-world applications are indicated by this icon.

There is a glossary at the back of the book. Glossary terms are **purple**.

These features are designed to promote the IB's inquiry-based approach, in which mathematics is not seen as a collection of facts to be learned, but a set of skills to be developed.

About the authors

The authors are all Cambridge University graduates and have a wide range of expertise in pure mathematics and in applications of mathematics, including economics, epidemiology, linguistics, philosophy and natural sciences.

Between them they have considerable experience of teaching IB Diploma Mathematics at Standard and Higher Level, and two of them currently teach at the University of Cambridge.

1 Exponents and logarithms

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.
- Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

In this chapter you will learn…

- how to extend the laws of exponents to general rational exponents
- how to use the laws of logarithms
- how to find the sum to infinity of a geometric series
- how to interpret graphs with logarithmic scales
- how to linearize data to infer the parameters of model.

CONCEPTS

The following concepts will be addressed in this chapter:

- Numbers and formulae can appear in different but **equivalent** forms, or **representations**, which can help us to establish identities.
- **Patterns** in numbers inform the development of algebraic tools that can be applied to find unknowns.
- **Generalization** provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.

LEARNER PROFILE – to follow

Text to be inserted at proof stage.

■ **Figure 1.1** Why are logarithms and exponential functions used to describe these phenomena?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Simplify the following:

a $3x^2y^4 \times 5x^7y$ **b** $\frac{8c^2d^3}{2c^3d}$ **c** $(3a^4b^{-2})^2$

2 Find the value of y for which $\log_{10}(y) = 2$.

3 Solve the equation $3^{x-5} = \left(\frac{1}{9}\right)^x$.

4 Solve the equation $e^x = 11$.

5 Find the 5th term of the geometric sequence with first term 4 and common ratio −2.

6 Use technology to find the Pearson's product-moment correlation coefficient and the equation of the y-on-x regression line for the data in the table.

x	1	2	2	5
y	2	6	4	8

You have already seen how the laws of exponents allow you to manipulate exponential expressions, and how this can be useful for solving some types of exponential equation.

In the same way, it is useful to have some laws of logarithms, which will enable you to solve some more complicated exponential equations and equations involving different logarithm terms. It should be no surprise, given the relationship between exponents and logarithms, that these laws of logarithms follow from the laws of indices.

Laws of logarithms and indices can also be applied to extending your knowledge of geometric series and used to turn graphs of functions in straight lines, thereby making it easier to estimate parameters.

Starter Activity

Look at the pictures in Figure 1.1. Investigate how exponential and logarithmic functions can be used to measure or model these different phenomena.

Now look at this problem:

By trying different positive values of x and y, suggest expressions for the following in terms of $\ln x$ and $\ln y$:

a $\ln(xy)$ **b** $\ln\left(\frac{x}{y}\right)$ **c** $\ln(x^y)$

Do your suggested relationships work for $\log_{10}$ as well?

1A Laws of exponents with rational exponents

In order to extend the laws of exponents for integer exponents that you already know to rational exponents that are not integers, you need a new law.

For a reminder of the laws of exponents for integer exponents see Section 1A of the Mathematics: Applications and interpretation SL book.

KEY POINT 1.1

- $a^{\frac{1}{n}} = \sqrt[n]{a}$

Proof 1.1

Explain why $a^{\frac{1}{n}} = \sqrt[n]{a}$.

We need a defining feature of $\sqrt[n]{a}$ .... $\sqrt[n]{a}$ is the number which equals a when raised to the exponent n.

Use the fact that $(x^a)^b \equiv x^{ab}$ .... We know that $\left(a^{\frac{1}{n}}\right)^n \equiv a^{\left(\frac{1}{n}\times n\right)} \equiv a^1$.

Therefore $a^{\frac{1}{n}}$ has the defining property of $\sqrt[n]{a}$.

CONCEPTS – REPRESENTATION

You might ask why writing the same thing in a different notation has any benefit. Using an exponent **representation** of $\sqrt[n]{a}$ has a distinct advantage as it allows us to use the laws of exponents on these expressions.

WORKED EXAMPLE 1.1

Evaluate $16^{\frac{1}{4}}$.

Use $a^{\frac{1}{n}} = \sqrt[n]{a}$ .... $16^{\frac{1}{4}} = \sqrt[4]{16}$

$= 2$

Combining the law in Key Point 1.1 with the law that states that $(a^m)^n = a^{mn}$ allows us to cope with any rational exponent.

WORKED EXAMPLE 1.2

Evaluate $27^{-\frac{2}{3}}$.

Use $(a^m)^n = a^{mn}$ to split the exponent: $27^{-\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^{-2}$

Remember that a negative exponent turns into 1 divided by the same expression with a positive exponent. $27^{\frac{1}{3}} = \sqrt[3]{27}$: $= \frac{1}{(\sqrt[3]{27})^2}$

You should know small perfect squares and cubes. You can recognize that $\sqrt[3]{27} = 3$: $= \frac{1}{3^2} = \frac{1}{9}$

You will also need to be able use this new law in an algebraic context.

WORKED EXAMPLE 1.3

Write $\frac{4x}{\sqrt[3]{x}}$ in the form kx^n.

Use $a^{\frac{1}{n}} = \sqrt[n]{a}$ on the denominator: $\frac{4x}{\sqrt[3]{x}} = \frac{4x}{x^{\frac{1}{3}}}$

Then use $\frac{a^m}{a^n} = a^{m-n}$: $= 4x^{1-\frac{1}{3}}$

$= 4x^{\frac{2}{3}}$

Exercise 1A

Although in the exam you will have a calculator, to develop understanding we recommend trying this whole exercise without a calculator.

For questions 1 to 8, use the method demonstrated in Worked Example 1.1 to evaluate without a calculator:

1 a $8^{\frac{1}{3}}$ b $27^{\frac{1}{3}}$

2 a $64^{\frac{1}{4}}$ b $625^{\frac{1}{4}}$

3 a $49^{\frac{1}{2}}$ b $25^{\frac{1}{2}}$

For questions 4 to 8, use the methods demonstrated in Worked Example 1.2 to evaluate without a calculator:

4 a $8^{\frac{2}{3}}$ b $16^{\frac{3}{4}}$

5 a $625^{\frac{3}{4}}$ b $125^{\frac{2}{3}}$

6 a $100^{-\frac{1}{2}}$ b $1000^{-\frac{1}{3}}$

7 a $8^{-\frac{2}{3}}$ b $27^{-\frac{2}{3}}$

8 a $32^{-\frac{2}{5}}$ b $100\,000^{-\frac{3}{5}}$

For questions 9 to 13 use the methods demonstrated in Worked Example 1.3 to simplify each expression, giving your answers in the form ax^p.

9 a $x^2\sqrt{x}$ b $x\sqrt[3]{x}$

10 a $\dfrac{x^2}{\sqrt[3]{x}}$ b $\dfrac{x^2}{\sqrt{x}}$

11 a $\dfrac{4\sqrt{x}}{x^2}$ b $\dfrac{5\sqrt{x}}{x^3}$

12 a $\dfrac{x^2}{5\sqrt{x}}$ b $\dfrac{x}{3\sqrt[3]{x}}$

13 a $\sqrt[3]{x}\sqrt{x}$ b $x\sqrt[3]{x^2}$

14 Find the exact value of $8^{-\frac{4}{3}}$.

15 Find the exact value of $\left(\dfrac{1}{4}\right)^{-\frac{1}{2}}$.

16 Find the exact value of $\left(\dfrac{4}{9}\right)^{\frac{3}{2}}$.

17 Write $\sqrt[3]{x^2}\times\sqrt[4]{x}$ in the form x^p.

18 Write $\dfrac{3}{\sqrt[3]{x}}+2\sqrt{x}$ in the form ax^p+bx^q.

19 Write $\dfrac{1}{3\sqrt{x^3}}$ in the form ax^p.

20 Solve the equation $x^{\frac{3}{2}}=\dfrac{1}{8}$.

21 Solve the equation $x^{-\frac{1}{2}}=\dfrac{2}{5}$.

22 Write in the form x^p: $\dfrac{x\sqrt{x}}{\sqrt[3]{x}}$

23 Write in the form x^p: $\dfrac{x}{x^2\sqrt{x}}$

24 Write $(x\times\sqrt[3]{x})^2$ in the form x^k.

25 Write $\left(\dfrac{1}{2\sqrt{x}}\right)^3$ in the form ax^p.

26 Write in the form x^a+x^b: $\dfrac{x^2+\sqrt{x}}{x\sqrt{x}}$

27 Write in the form x^a-x^b: $\dfrac{(x+\sqrt{x})(x-\sqrt{x})}{\sqrt{x}}$

28 Write in the form ax^k: $\dfrac{1}{3x\sqrt{x}}$

29 Write in the form ax^p+bx^q: $\dfrac{x^2+3\sqrt{x}}{2x}$

30 Given that $y=2\sqrt[3]{x^2}$, write y^4 in the form ax^k.

31 Given that $y=27\sqrt{x}$, write $\sqrt[3]{y}$ in the form ax^k.

32 Given that $\sqrt{x}=\sqrt[3]{y}$, write y in the form x^k.

33 Given that $y=\dfrac{2}{3\sqrt{x}}$, write y^3 in the form ax^k.

34 Solve $(\sqrt{3})^x=9^{x-1}$.

35 Solve $(\sqrt{2})^x=4^{x+2}$.

36 Solve $(\sqrt[3]{2})^{2x}=8^{x+1}$.

37 Solve $(\sqrt[3]{3})^{4x}=9^{x-3}$.

38 Given that $y=x\sqrt{x}$, express x in terms of y.

39 Solve the equation $x^{\frac{2}{3}}=9$.

40 Solve the equation $\sqrt{x}=2\sqrt[3]{x}$.

1B Logarithms

You met logarithms in the Mathematics: Applications and interpretation SL book. The logarithm to the base a of b is defined as the power a must be raised to if you want the result to be b. We can write this as:

$b=a^x$ is equivalent to $\log_a b=x$

You will see in Chapter 4 that this means that logarithms and exponents are inverse functions – one undoes the other.

WORKED EXAMPLE 1.4

Find the exact value of y if $\log_{10}(y-3) = -1$.

Use $b = a^x$ is equivalent to $\log_a b = x$ .... $y - 3 = 10^{-1} = \frac{1}{10}$

$y = 3.1$

Remember that logarithms to the base e are given a special notation, $\ln x$.

Tip

Remember that most equations like this can actually be solved graphically, using technology. However, solving them manually can aid understanding.

WORKED EXAMPLE 1.5

Solve $2 + \ln x = 0$.

Isolate the $\ln x$ term .... $\ln x = -2$

Use $b = a^x$ is equivalent to $\log_a b = x$. Remember that $\ln x$ is equivalent to $\log_e x$ .... $x = e^{-2} \approx 0.135$

Laws of logarithms

Tip

Be careful not to apply similar looking rules of logarithms. For example, many students claim that $\log(x + y) = \log x + \log y$ or that $\log(x + y) = \log x \times \log y$. Neither of these are true in general.

The laws of exponents lead to a set of laws of logarithms.

KEY POINT 1.2

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_b x - \log_a y$
- $\log_a x^m = m \log_a x$

where $a, x, y > 0$

CONCEPTS – PATTERNS

Many rules like this are explored and discovered by systematically looking at **patterns** in numbers.

For example:

$\log_{10} 2 = 0.30103$ $\log_{10} 20 = 1.30103$ $\log_{10} 200 = 2.30103$ $\log_{10} 2000 = 3.30103$

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The first law of logarithms is proved here. The others are proved similarly from the corresponding law of exponents.

Proof 1.2

Prove that $\log_a xy = \log_a x + \log_a y$.

Start by using the related law of exponents $a^m \times a^n = a^{m+n}$ with $m = \log_a x$ and $n = \log_a y$

$$a^{\log_a x} \times a^{\log_a y} = a^{\log_a x + \log_a y}$$

Use $a^{\log_a x} = x$ on both terms of the product on the LHS

$$xy = a^{\log_a x + \log_a y}$$

Take $\log_a$ of both sides

$$\log_a xy = \log_a a^{\log_a x + \log_a y}$$

Use $\log_a a^x = x$ on the RHS

$$\log_a xy = \log_a x + \log_a y$$

CONCEPTS – EQUIVALENCE

The ability to go easily between representing equations using logs and using exponents allows us to turn our old rules into **equivalent** new rules. This is a very common and powerful technique in many areas of mathematics – for example looking at how rules in differentiation apply to integration.

WORKED EXAMPLE 1.6

If $p = \log_a$ and $q = \log_b$, express $\log\left(\frac{a^3}{b}\right)$ in terms of p and q.

Use $\log_a \frac{x}{y} = \log_a x - \log_a y$. The question does not specify what base to use, as it actually does not matter.

$$\log\left(\frac{a^3}{b}\right) = \log a^3 - \log b$$

Then use $\log_a x^m = m\log_a x$ on the first term

$$= 3\log a - \log b$$

Now replace $\log_a$ with p and $\log_b$ with q

$$= 3p - q$$

One common application of the laws of logarithms is in solving log equations. The usual method is to combine all log terms into one.

WORKED EXAMPLE 1.7

Solve

$$\log_{10}(3x-2)-\log_{10}(x-4)=1$$

$$\log_{10}(3x-2)-\log_{10}(x-4)=1$$

Combine the log terms using $\log_a x-\log_a y=\log_a \frac{x}{y}$

$$\log_{10}\left(\frac{3x-2}{x-4}\right)=1$$

Remove the log using $\log_a b = x$ is equivalent to $b=a^x$

$$\frac{3x-2}{x-4}=10^1$$

$$3x-2=10(x-4)$$

$$3x-2=10x-40$$

Solve for x

$$38=7x$$

$$x=\frac{38}{7}\approx 5.43$$

Be the Examiner 1.1

Solve $\log_{10}(x+10)+\log_{10}2=2$

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\log_{10}(x+12)=2$	$\log_{10}(2x+10)=2$	$\log_{10}(2x+20)=2$
$x+12=100$	$2x+10=100$	$2x+20=100$
$x=88$	$x=45$	$x=40$

TOOLKIT: Problem Solving

In the exam you will only be asked about logs to the base of 10 and *e*. However, you might find solving the following equations helps you to really understand logarithms:

a $x=\log_3 27$ **b** $4=\log_2 x$ **c** $3=\log_x 64$

Solving exponential equations

One common use of logarithms is to solve exponential equations – these are equations where the unknowns are in the powers. The technique is to take logarithms of both sides (to any convenient base), apply the laws of logarithms to turn the equation into a linear equation and then solve for x.

WORKED EXAMPLE 1.8

Solve the equation

$7^{x-2} = 5^{x+3}$ giving your answer in terms of natural logarithms.

Take natural logs of both sides ...... $\ln(7^{x-2}) = \ln(5^{x+3})$

Use $\ln a^m = m \ln a$ ...... $(x-2)\ln 7 = (x+3)\ln 5$

Expand the brackets ...... $x\ln 7 - 2\ln 7 = x\ln 5 + 3\ln 5$

Group the x terms and the number terms (remember that $\ln 5$ and $\ln 7$ are just numbers) ...... $x\ln 7 - x\ln 5 = 3\ln 5 + 2\ln 7$

$x(\ln 7 - \ln 5) = 3\ln 5 + 2\ln 7$

Factorize the left-hand side and divide ...... $x = \dfrac{3\ln 5 + 2\ln 7}{\ln 7 - \ln 5}$

Tip

The final answer can be written in a different form, by using laws of logarithms on the top and the bottom:

$$x = \frac{\ln(5^3 \times 7^2)}{\ln(7 \div 5)} = \frac{\ln(6125)}{\ln(1.4)}.$$

Exercise 1B

For questions 1 to 4, use the method demonstrated in Worked Example 1.4 to solve the equations.

1 a $\log_{10} x = -2$
b $\log_{10} x = -1$

2 a $\log_{10}(y-1) = -1$
b $\log_{10}(y-2) = -2$

3 a $\log_{10}(x+2) = 2$
b $\log_{10}(x+3) = 3$

4 a $\log_{10}(t-1) = 0$
b $\log_{10}(r+3) = 0$

For questions 5 to 7 use the method demonstrated in Worked Example 1.5 to solve the equations

5 a $\ln x = 2$
b $\ln x = 5$

6 a $2\ln x - 1 = 0$
b $3\ln x + 1 = 0$

7 a $\ln(2x+4) = 2$
b $\ln(3x-1) = 4$

For questions 8 to 11 use the methods demonstrated in Worked Example 11.6. Write each given expression in terms of p and q, where $p = \log_{10} a$ and $q = \log_{10} b$.

8 a $\log_{10}\left(\dfrac{a^2}{b}\right)$
b $\log_{10}\left(\dfrac{b^3}{a}\right)$

9 a $\log_{10}\left(\dfrac{a^2}{b^3}\right)$
b $\log_{10}\left(\dfrac{b^4}{a^2}\right)$

10 a $\log_{10}\sqrt{a^3 b}$
b $\log_{10}\sqrt{a^4 b^3}$

11 a $\log_{10}\left(\dfrac{100a}{b^2}\right)$
b $\log_{10}\left(\dfrac{10a^2}{b^5}\right)$

For questions 12 to 15 use the methods demonstrated in Worked Example 1.7 to solve the equations, giving your answers in an exact form

12 a $\log_{10} x + \log_{10} 2 = 3$
b $\log_{10} x + \log_{10} 5 = 1$

13 a $\log_{10}(x+3) + \log_{10} 2 = 2$
b $\log_{10}(x-1) + \log_{10} 4 = 1$

14 a $\log_{10}(x+1) - \log_{10}(x-2) = 1$
b $\log_{10}(x+1) - \log_{10}(x-1) = 2$

15 a $\ln(x-3) - \ln(x+5) = 4$
b $\ln(x+2) - \ln(x-1) = 3$

For questions 16 and 17 use the method demonstrated in Worked Example 1.8 to solve the equations, giving your answer in terms of natural logarithms.

16 a $3^{x-2} = 2^{x+1}$
b $3^{x-1} = 2^{x+2}$

17 a $7^{2x-5} = 2^{x+3}$
b $7^{3x+1} = 2^{x+8}$

18 Given that $x = \log_{10} a$, $y = \log_{10} b$ and $z = \log_{10} c$, write the following in terms of x, y and z:

a $\log_{10}(ab^4)$ b $\log_{10}\left(\frac{a^2b}{c^5}\right)$ c $\log_{10}(10a^2b^3)$

19 Given that $x = \log_{10} a$, $y = \log_{10} b$ and $z = \log_{10} c$, write the following in terms of x, y and z:

a $\log_{10}(100\sqrt{a})$ c $\log_{10}\left(\frac{b}{10c^5}\right)$

20 Write $2\ln a + 6\ln b$ as a single logarithm.

21 Write $\frac{1}{3}\ln x - \frac{1}{2}\ln y$ as a single logarithm.

22 Solve $\log_{10}(x+3) = 3$.

23 Solve the equation $\log_{10}(2x-4) = 1$.

24 Use logarithms to solve these equations:

a $5^x = 10$ b $2 \times 3^x + 6 = 20$

25 Solve the equation $3 \times 1.1^x = 20$.

26 a Find the exact value of $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$.

b Solve the equation $\log_x 27 = -\frac{1}{3}$.

27 Solve the equation $\log_x 32 = 5$.

28 Solve the equation $\log_x 64 = 3$.

29 Given that $5 \times 6^x = 12 \times 3^x$,

a write down the exact value of 2^x.

b hence find the value of x.

30 Solve the equation $8^{3x+1} = 4^{x-3}$.

31 Solve the equation $5^{2x+3} = 9^{x-5}$, giving your answer in terms of natural logarithms.

32 The radioactivity (R) of a substance after a time t days is modelled by $R = 10 \times 0.9^t$

a Find the initial (i.e. $t = 0$) radioactivity.

b Find the time taken for the radioactivity to fall to half of its original value.

33 The population of bacteria (B) at time t hours after being added to an agar dish is modelled by $B = 1000 \times 1.1^t$

a Find the number of bacteria **i** initially **ii** after 2 hours.

b Find an expression for the time it takes to reach 2000. Use technology to evaluate this expression.

34 The population of penguins (P) after t years on two islands is modelled by:

First island: $P = 200 \times 1.1^t$

Second island: $P = 100 \times 1.2^t$

How many years are required before the population of penguins on both islands is equal?

35 Solve the simultaneous equations

$\log_{10} x + \log_{10} y = 3$ $\log_{10} x - 2\log_{10} y = 0$

36 Solve the simultaneous equations

$\log_{10} x + \ln y = 1$ $\log_{10} x^2 + \ln y^3 = 4$

37 Solve the equation $2^{5-3x} = 3^{2x-1}$, giving your answer in the form $\frac{\ln p}{\ln q}$, where p and q are integers.

38 Moore's law states that the density of transistors on an integrated circuit doubles every 2 years. Find the time taken for the density to multiply by 10.

39 a If $\log_a(x^2) = b$ find the product of all possible values of x.

b If $(\log_a x)^2 = b$ find the product of all possible values of x.

1C Sum of infinite convergent geometric sequences

You met the sum of a geometric sequence in Section 2B of the Mathematics: Applications and interpretation SL book.

You met the idea of limits in Section 9A of the Mathematics: Applications and interpretation SL book.

Geometric sequences are closely related to exponential functions. The main difference is that the domain of exponential functions can be all real numbers, but the domain of geometric sequences is normally restricted to positive integers.

You know that you can find the sum of the first n terms of a geometric sequence using the formula $S_n = \frac{a(1-r^n)}{1-r}$. Sometimes this sum will just increase (or decrease if negative) the more terms you add.

However, if r is between 1 and –1 then as n gets very large, r^n tends towards 0.

As a result, $\frac{a(1-r^n)}{1-r}$ tends towards $\frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

So, in this situation the sum of infinitely many terms converges to a finite limit – this is called the **sum to infinity**, S_∞, of the geometric sequence.

KEY POINT 1.3

For a geometric sequence with common ratio r,

- $S_\infty = \frac{u_1}{1-r}$ if $|r| < 1$

WORKED EXAMPLE 1.9

Find the value of the infinite geometric series

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$$

This is a geometric series with first term 2 and common ratio $\frac{2}{3}$

$u_1 = 2,\ r = \frac{2}{3}$

Use $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{2}{1-\frac{2}{3}} = 6$$

You are the Researcher

The ancient Greeks had some real difficulties working with limits of series. Some of them thought incorrectly that if each term in the sum got smaller and smaller, then the series would reach a limit. However, this is not always true. You might like to explore the Harmonic series, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$ and see why that does not converge.

There are many tests that mathematicians use to decide if a series will converge, for example the ratio test. You might like to learn more about these and how they work.

WORKED EXAMPLE 1.10

The geometric series $(x+4)+(x+4)^2+(x+4)^3+\dots$ converges.

Find the range of possible values of x.

State the common ratio .... $r = x + 4$

Since the series converges,

$|x+4| < 1$

You know that $|r| < 1$ .... $-1 < x + 4 < 1$

$-5 < x < -3$

Be the Examiner 1.2

Find the sum to infinity of the geometric series,

$$\frac{1}{2}-\frac{3}{4}+\frac{9}{8}-\frac{27}{16}+\dots$$

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$S_\infty = \dfrac{1}{1-\frac{1}{2}}$ $= 2$	$S_\infty = \dfrac{\frac{1}{2}}{1-\left(-\frac{3}{2}\right)}$ $= \dfrac{1}{5}$	$r = -\frac{3}{4} \div \frac{1}{2} = -\frac{8}{3}$ so it does not exist.

TOOLKIT: Problem Solving

You may not have realized it, but you have already met infinite geometric series in your previous work. Explain how you can find the following sum using methods from your prior learning. Confirm the answer by using the formula for the geometric series.

$$\sum_{r=1}^{\infty} \frac{3}{10^r}$$

What happens to your argument when applied to the sum below?

$$\sum_{r=1}^{\infty} \frac{9}{10^r}$$

Exercise 1C

For questions 1 to 5, use the method demonstrated in Worked Example 1.8 to find the sum of the infinite geometric series.

1 a $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots$

b $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

2 a $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\dots$

b $\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\dots$

3 a $4+1+\frac{1}{4}+\dots$

b $6+2+\frac{2}{3}+\dots$

4 a $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$

b $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\ldots$

5 a $15-9+\frac{27}{5}-\frac{81}{25}+\ldots$

b $16-12+9-\frac{27}{4}+\ldots$

For questions 6 to 10 use the method demonstrated in Worked Example 1.9 to find the range of values of x for which the infinite geometric series converges.

6 a $(x-2)+(x-2)^2+(x-2)^3+\ldots$

b $(x+3)+(x+3)^2+(x+3)^3+\ldots$

7 a $1+2x+4x^2+\ldots$

b $1+3x+9x^2+\ldots$

8 a $1+\frac{x}{2}+\frac{x^2}{4}+\ldots$

b $1+\frac{x}{5}+\frac{x^2}{25}+\ldots$

9 a $(x+4)-(x+4)^2+(x+4)^3-\ldots$

b $(x-1)-(x-1)^2+(x-1)^3-\ldots$

10 a $1-\frac{3x}{2}+\frac{9x^2}{4}-\ldots$

b $1-\frac{4x}{3}+\frac{16x^2}{9}-\ldots$

11 A geometric series has first term 3 and common ratio $\frac{1}{4}$. Find the sum to infinity of the series.

12 Find the sum to infinity of the geometric series with first term 5 and common ratio $-\frac{1}{4}$.

13 Find the sum to infinity of the series $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\ldots$

14 An infinite geometric series has the first term 8 and its sum to infinity is 6. Find the common ratio of the series.

15 The first term of a geometric series is 3. Given that the sum to infinity of the series is 4, find the value of the common ratio.

16 The sum to infinity of a geometric series is 3 and the common ratio is $\frac{1}{3}$. Find the first three terms of the series.

17 The second term of a geometric series is 2 and the sum to infinity is 9. Find two possible values of the common ratio.

18 A geometric series is given by $\sum_{r=0}^{\infty}\left(\frac{2}{5}\right)^r$.

a Write down the first three terms of the series.

b Find the sum of the series.

19 Evaluate $\sum_{r=0}^{\infty}\frac{2}{3^r}$.

20 a Find the range of values of x for which the geometric series $3-\frac{x}{3}+\frac{x^2}{27}-\frac{x^3}{243}+\ldots$ converges.

b Find the sum of the series when $x=-2$.

21 A geometric series is given by $5+5(x-3)+5(x-3)^2+\ldots$

a Find the range of values of x for which the series converges.

b Find the expression, in terms of x, for the sum of the series.

22 For the geometric series $2+4x+8x^2+\ldots$

a Find the range of values of x for which the series converges.

b Find the expression, in terms of x, for the sum to infinity of the series.

23 The second term of a geometric series is $-\frac{6}{5}$ and the sum to infinity is 5. Find the first term of the series.

24 Given that x is a positive number,

a Find the range of values of x for which the geometric series $x+4x^3+16x^5+\ldots$ converges.

b Find an expression, in terms of x, for the sum to infinity of the series.

25 a Find the value of x such that $\sum_{r=0}^{\infty}\frac{x^{r+1}}{2^r}=3$.

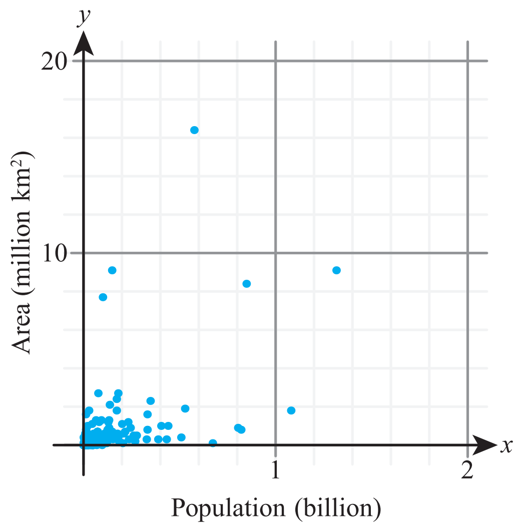
b Find the range of values of x for which the series converges.

26 An infinite geometric series has sum to infinity of 27 and sum of the first three terms equal to 19. Find the first term.

1D Using logarithmic scales on graphs

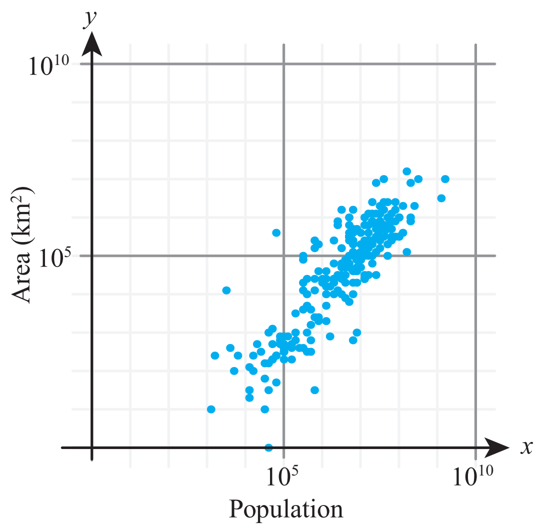
Scaling very large or very small numbers using logarithms

Sometimes you might be interested in displaying data which covers a very wide range of values. For example, the areas and populations of different countries. A raw plot of this data is not very helpful:



This is because some of the largest countries – China, India and Russia – are so much larger than the other countries that they mainly get squashed into the bottom left corner, so no detail can be seen.

A much better to way to represent data like this is to use a logarithmic scale. This is where the logs of the original numbers are plotted instead (although sometimes, as here, the original numbers are still plotted on the axes).

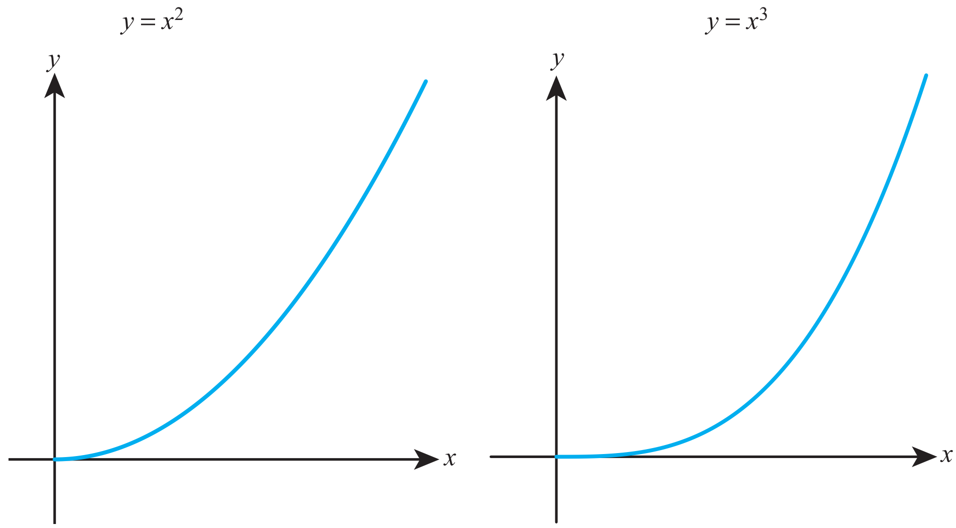


TOOLKIT: Modelling

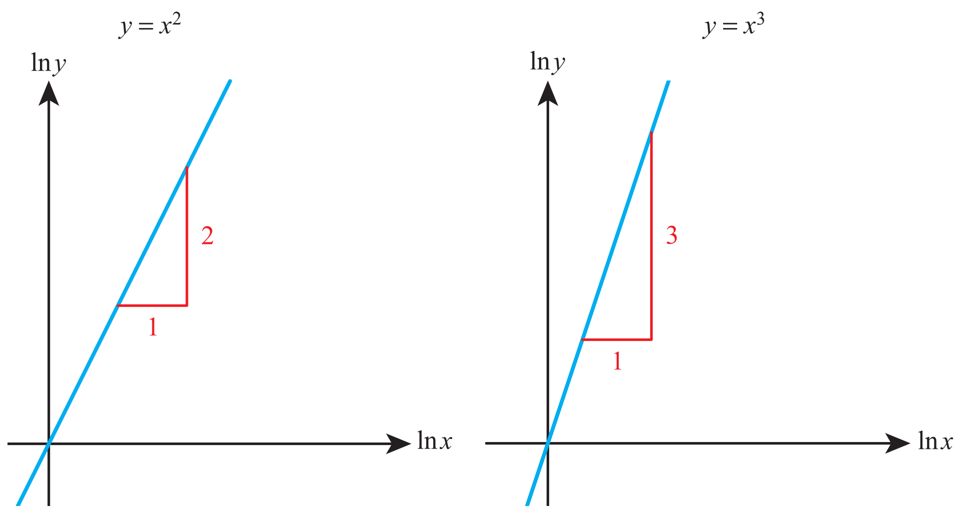
See if you can think of any other data which might cover several orders of magnitude. Try to find this data and plot it with and without logarithmic scales.

Linearizing data using logarithms

If you are just looking at positive values of x, it can be very tricky to visually tell the difference between the graphs of $y = x^2$ and $y = x^3$.



However, if we plot the logarithm of y against the logarithm of x we find something very interesting:



The graphs become straight lines through the origin with different gradients. To explain this, look at what happens if we investigate the general power relationship $y = ax^n$. Taking logs of both sides we find

$$\log y = \log(ax^n)$$

Notice that it does not actually matter what base we use. Applying the laws of logarithms:

$$\log y = n \log x + \log a$$

This has the same form as the straight line graph:

$Y = mX + c$

KEY POINT 1.4

If $y = ax^n$ the graph of $\log y$ against $\log x$ will be a straight line with gradient n and y-intercept $\log a$.

This is called **linearizing** the function $y = ax^n$. Plotting $\log y$ against $\log x$ is called creating a **log–log graph**.

WORKED EXAMPLE 1.11

Linearize the relationship $y = 3x^4$ and describe the resulting graph.

Take logs of both sides. It does not matter what base you use

$\ln y = \ln(3x^4)$

Apply the laws of logarithms

$= \ln 3 + \ln x^4$

$= \ln 3 + 4\ln x$

So the graph of $\ln y$ against $\ln x$ will be a straight line with gradient 4 and y-intercept $\ln 3$.

If a function obeys an exponential law, $y = ka^x$, then taking logs of both sides results in:

$\log y = (\log a)x + \log k$

This also has the same form as the straight line graph:

$Y = mX + c$

KEY POINT 1.5

If $y = ka^x$ the graph of $\log y$ against x will be a straight line with gradient $\log a$ and y-intercept $\log k$.

Plotting $\log y$ against x is called creating a **semi-log graph**.

WORKED EXAMPLE 1.12

Linearize the relationship $y = 2 \times 4^x$ and describe resulting graph.

Take logs of both sides. It does not matter what base you use

$\log_{10} y = \log_{10}(2 \times 4^x)$

Apply the laws of logarithms

$= \log_{10}(2) + \log_{10}(4^x)$

$= \log_{10}(2) + x\log_{10}(4)$

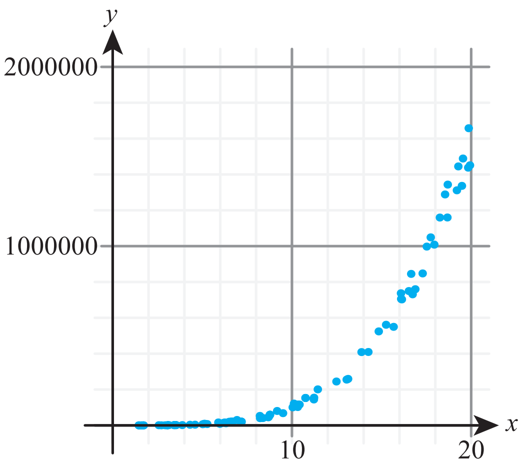
So the graph of $\log_{10} y$ against x will be a straight line with gradient $\ln 4$ and intercept $\ln 2$.

Interpretation of log–log and semi-log graphs

Commonly, log–log and semi-log graphs are not used on perfect functions. They are applied to data with the aim of inferring which sort of relationship might exist.

WORKED EXAMPLE 1.13

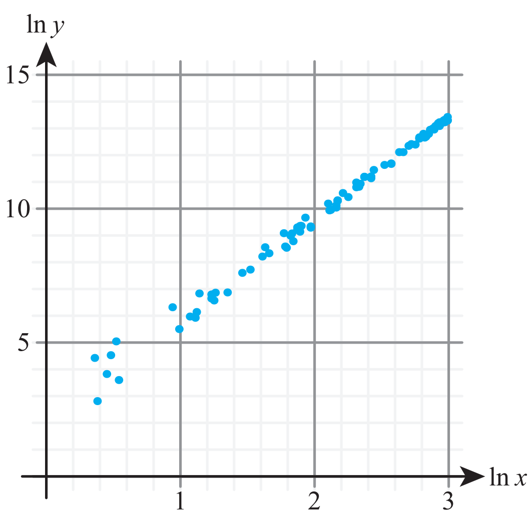
The graph below shows some collected data.

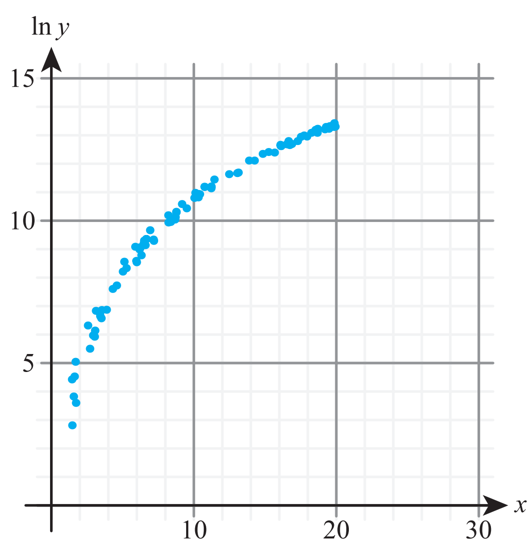


Tip

You can use your knowledge of linear regression and correlation to find the line of best fit in a more rigorous way.

The log–log and semi-log plots are shown below.



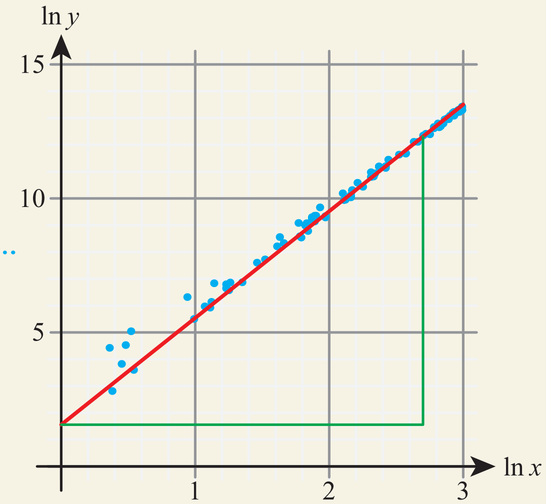


Determine the relationship between x and y.

Decide which of the graphs is plausibly linear. It does not have to be perfect.

The log–log plot looks closer to a straight line than the semi-log plot, so it is a power relationship.

Add a line of best fit by eye to the graph



Find the gradient and intercept of the line of best fit

$$\text{gradient} \approx \frac{12.4 - 1.6}{2.7 - 0} = 4$$

y-intercept ≈ 1.6

Use Key point 1.4. Since the intercept is $\ln a$ then $a = e^{\text{intercept}}$

So $y = ax^n$ with $n \approx 4$ and $a \approx e^{1.6} \approx 4.95$

Exercise 1D

For questions 1 to 4, use the methods demonstrated in Worked Example 1.10 to linearize the given functions and describe the resulting graph.

1 **a** $y = x^4$
b $y = x^5$

2 **a** $y = 3x^4$
b $y = 5x^2$

3 **a** $y = 4\sqrt{x}$
b $y = 2\sqrt[3]{x}$

4 **a** $y = \frac{1}{x}$
b $y = \frac{2}{\sqrt{x}}$

For questions 5 to 8, use the methods demonstrated in Worked Example 1.11 to linearize the given functions and describe the resulting graph.

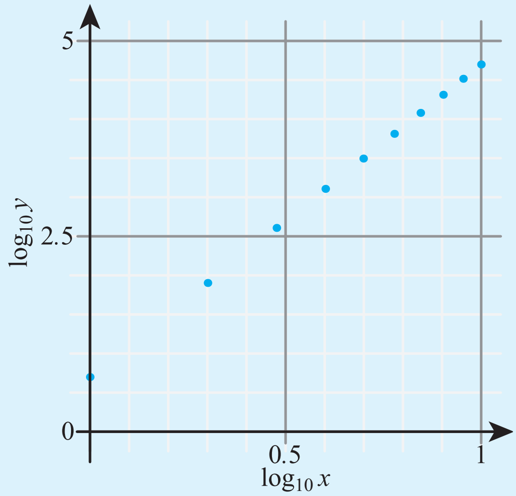
5 **a** $y = 2^x$
b $y = 5^x$

6 **a** $y = 3 \times 2^x$
b $y = 2 \times 3^x$

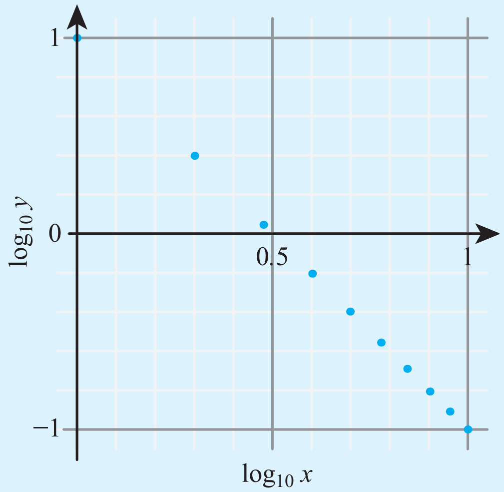
7 **a** $y = \frac{4}{7^x}$
b $y = \frac{6}{4^x}$

8 **a** $y = 2e^{3x}$
b $y = 10e^{-x}$

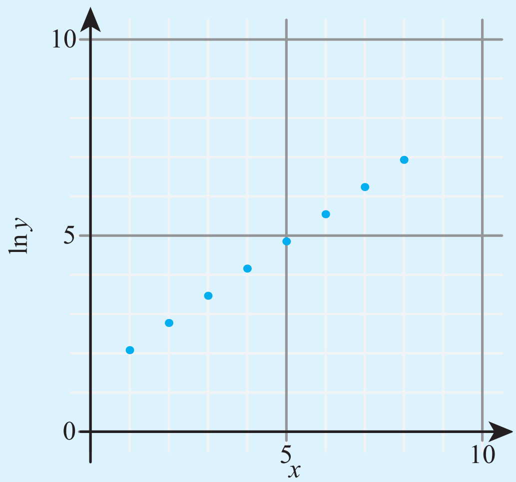
9 The graph below shows a log–log graph. Use this to suggest the relationship between x and y.



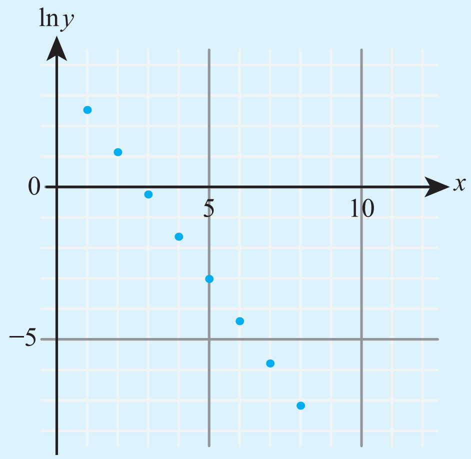
10 The graph below shows a log–log graph. Use this to suggest the relationship between x and y.



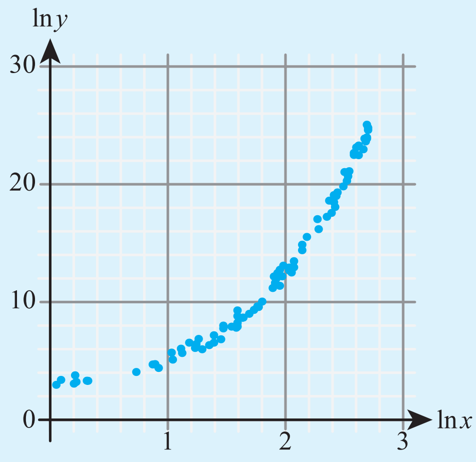
11 The graph below shows a semi-log graph. Use this to suggest the relationship between x and y.

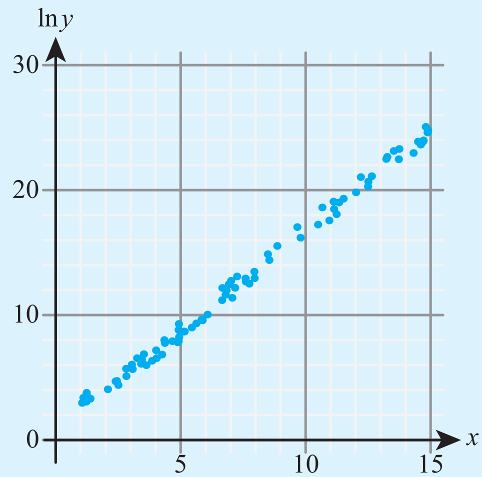


12 The graph below shows a semi-log graph. Use this to suggest the relationship between x and y.



13 The scatterplots below show a semi-log and log–log plot for a set of data.
Use these plots to suggest a relationship between x and y, giving the values of any parameters to 1 significant figure.





14 The amount of a radioactive substance, X grams, at a time t minutes after it is created is modelled by $X = X_0 e^{-kt}$.

a Linearize this relationship.

b The following measurements were taken:

t	1	2	3	4
X	12	7	4	3

Use linear regression to estimate the value of X_0 and k.

c Use your model to estimate the time taken for the amount of substance to halve.

15 The force, F Newtons, between two magnets a distance d metres apart is modelled by the equation $F = kd^n$.

a Linearize this relationship.

b The following measurements were taken:

d	1	2	3	4
F	12	7	4	3

Use linear regression to find the values of k and n, giving your answer to 2 significant figures.

16 The following data show the length of railway added in the United States by year:

Year	1843	1844	1845	1846	1847	1848	1849	1850
Miles added	159	192	256	297	668	398	1369	1656

A model is created of the form $y = ab^m$ where y is the number of years after 1842 and m is the number of miles added.

a Use linear regression to estimate the values of a and b.

b What is the average percentage increase in the number of miles of railway added?

c Hence estimate the number of miles added in 1860.

17 The data in the table shows the average distance from the sun, d million km, and year length, y days, of several planets in the solar system.

Planet	d	y
Mercury	58	88
Venus	108	225
Earth	150	365
Mars	228	687
Jupiter	779	4380
Saturn	1434	10 585

a Use linear regression to form a model of the form $y = ad^n$, giving the parameters to one decimal place.

b Hence estimate the year length, to the nearest earth year, of Uranus which is 2871 million km from the sun.

18 Lotka's law states that the number of authors making a contribution to x published scientific papers is proportional to $\frac{1}{x^n}$ where n is an integer.

The data in the table are taken from a scientific journal.

x	1	2	3	4
Frequency	89	28	10	6

Use linear regression to find the integer value of n.

19 Becky records the population (P) in millions of six cities and the rank (R) of its size within Becky's country. Her results are:

R	1	2	3	4	5	6
P	6.5	3.4	2.1	1.6	1.2	1.1

a Using the natural logarithm find the value of Pearson's product-moment correlation coefficient for;

i the log–log graph

ii the semi-log graph.

b Hence suggest a form for the relationship between R and P, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

You are the Researcher

The model that you found in part **b** is one actually observed in most countries. It is called the rank-size distribution. Investigate if this rule apples in your country.

20 The data below is taken from the 19th century story "Treasure Ireland" by Robert Louis Stevenson (Source: Oxford text archive). It shows the 16 most common words used along with their rank (R) and frequency (F).

Word	R	F
the	1	4355
and	2	2872
I	3	1748
a	4	1745
of	5	1665
to	6	1520
was	7	1135
in	8	968
he	9	901
that	10	835
you	11	824
had	12	738
it	13	706
his	14	651
as	15	623
my	16	618

From Oxford Text Archive, part of the Bodleian Libraries

a Using the natural logarithm find the value of Pearson's product-moment correlation coefficient for:

  i the log–log graph

  ii the semi-log graph.

b Hence suggest a form for the relationship between R and F, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

d Assuming that the least common word in Treasure Island is used 10 times, use your model to estimate how many *different* words are used in Treasure Island.

You are the Researcher

The model that you found in part **b** is one actually observed in most countries. It is called the rank-size distribution. Investigate if this rule apples in your country.

The relationship you found in **b** is called Zipf's law after the American linguist George Zipf (1902–1950), although it was also noticed by the French stenographer Jean-Baptiste Estoup (1868–1950) and the German physicist Felix Auerbach (1856-1933). Remarkably, it seems to apply across all known languages.

You are the Researcher

Word frequency analysis such as this is one of the tools used by forensic linguists to answer questions such as 'Did Shakespeare write all of his plays himself?', or 'Is a section of a piece of work plagiarised?'. It is also used by cryptographers (along with letter frequency analysis) to crack codes. You might want to find out more about how they use these techniques.

21 The variables x and y in the table below are thought to be modelled by a relationship of the form $y = ax^n$.

x	−3	−2	−1	1	2	3
y	−45.6	−19.8	−5.1	−4.7	−19.2	−44.2

Use linear regression to find the values of a and b.

22 Newton's law of cooling suggests that the temperature of a cooling body (T °C) at a time t hours will follow $T = Ae^{-kt} + c$.

A body is found at 12 noon. This point is chosen for $t = 0$. A forensic scientist takes the following measurements:

t	1	2	3	4
T	27	24.5	23	22

a Given that the background room temperature is assumed to be a constant 20 °C, linearize the given relationship and use linear regression to find A and k.

b Assuming that body temperature is normally 37 °C, use the model to estimate the time of death, giving your answer to the nearest 10 minutes.

23 The population of rabbits (P thousands) on an island at a time t years after they are introduced is modelled by the logistic function

$$P = \frac{10}{1 + Je^{-\beta t}}$$

a Find the value of P as t gets very large.

b Linearize the relationship.

c An ecologist finds the following measurements:

t	2	3	4	5
P	6	6.5	6.9	7.3

Use linear regression to estimate the value of β and J.

d Estimate how long it will take from the rabbit population being introduced for the rabbit population to reach 9000.

e Estimate the number of rabbits which were initially introduced to the island.

f Explain why your answers to **d** and **e** may not be accurate.

24 In biochemistry, the rate at which a reaction occurs, v, in the presence of an enzyme depends on the concentration of the reactant, $[S]$ according to: $v = \frac{v_{max}[S]}{K_m + [S]}$

where K_m is a constant reflecting the efficiency of the enzyme and v_{max} is the rate in an excess of reactant.

a What is the rate when $[S] = K_m$.

b Sketch the graph of v against $[S]$.

c Show that a plot of $\frac{1}{v}$ against $\frac{1}{[S]}$ linearizes the relationship, and state the gradient and y-intercept of the resulting line.

d Nasar measures the rate of reaction at different reactant concentrations. His results are shown below:

$[S]$	1	2	3	4
v	2	3	3.5	3.7

i Transform the data as suggested in part **c** and use least squares regression to find the equation of the regression line.

ii Hence estimate the values of v_{max} and K_m.

e i Show that a plot of v against $\frac{v}{[S]}$ is linear, and state the gradient and intercept.

ii Use Nasar's data to estimate the values of v_{max} and K_m using the linearization in **e i**.

Links to: Biology

The model of the rate of enzymatic reaction or 'enzyme kinetics' in question 24 is called Michaelis–Menten kinetics and it is very important in biology. The plot in part **c** is called a Lineweaver–Burk plot and the plot in part **e** is called an Eadie–Hofstee plot. You might wonder why we need two different plots. If the model was perfect and the data accurate then both plots would give the same answer; however neither of those conditions is true. The two different plots magnify different types of error. In modern biology both plots have been superseded by non-linear regression techniques, which you will meet in Chapter 10.

Checklist

- You should be able to extend the laws of exponents to general rational exponents:
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
- You should know the laws of logarithms:
  - $\log_a xy = \log_a x + \log_a y$
  - $\log_a \frac{x}{y} = \log_b x - \log_a y$
  - $\log_a x^m = m \log_a x$
- where $a, x, y > 0$.
- You should be able to find the sum of infinite geometric sequences:
- For a geometric sequence with common ratio r,

  $$S_\infty = \frac{u_1}{1-r} \quad \text{if } |r| < 1$$
- If $y = ax^n$ the graph of $\log y$ against $\log x$ will be a straight line with gradient n and y-intercept $\log a$.
- If $y = ka^x$ the graph of $\log y$ against x will be a straight line with gradient $\log a$ and y-intercept $\log k$.

Mixed Practice

1 **a** Find the exact value of $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$.

b Find $\log_{10}\left(\frac{1}{1000}\right)$.

2 Write $\ln 4 + 2\ln 3$ in the form $\ln k$.

3 Solve the equation $2 \times 3^{x-2} = 54$.

4 Use technology to solve $1.05^x = 2$.

5 Solve the equation $100^{x+1} = 10^{3x}$.

6 Find the value of x such that $\log_{10}(5x + 10) = 2$.

7 Find the exact solution of the equation $3\ln x + 2 = 2(\ln x - 1)$.

8 Given that $a = \log_{10} x$, $b = \log_{10} y$ and $c = \log_{10} z$, write the following in terms of a, b and c:

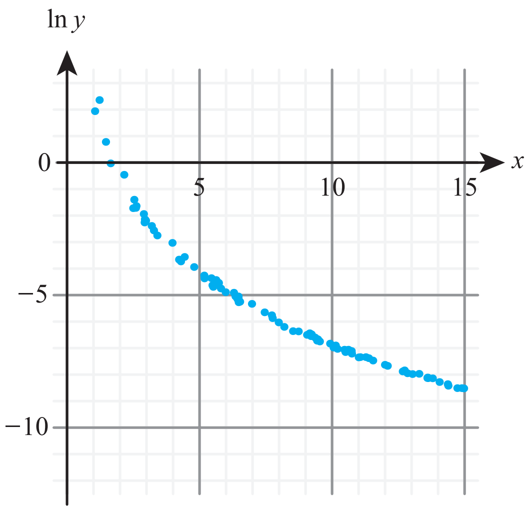
a $\log_{10}(x^2 y)$ **b** $\log_{10}\left(\frac{x}{yz^3}\right)$ **c** $\log_{10}(\sqrt{zx^3})$

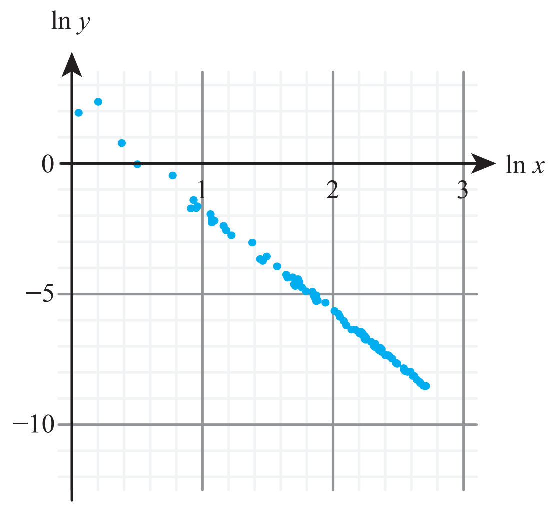
9 Find the sum of the infinite geometric series $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

10 A geometric series has first term 7 and common ratio $-\frac{5}{9}$. Find the sum to infinity of the series.

11 Find the first term of the geometric series with common ratio $\frac{3}{4}$ and sum to infinity 12.

12 The plots show a semi-log and log–log plot for a set of data.





Use these plots to suggest a relationship between x and y, giving the values of any parameters to 1 significant figure.

13 Given that $x = \log_{10} a$ and $y = \log_{10} b$, write in terms of x and y:

a $\log_{10}\left(\frac{a}{\sqrt{b}}\right)$ **b** $\log_{10}\left(\frac{a^2}{1000b^3}\right)$

14 Given that $x = \ln 2$ and $y = \ln 5$, write the following in terms of x and y:

a $\ln 10$ **b** $\ln 50$ **c** $\ln 0.08$

15 Write $3 + 2\log 5 - 2\log 2$ as a single logarithm.

16 Find the exact solution of the equation $3\ln x + \ln 8 = 5$, giving your answer in the form Ae^k where A and k are fractions.

17 Solve the equation $4^{3x+5} = 8^{x-1}$.

18 Solve the simultaneous equations:

$\log_{10} x + \log_{10} y = 5$

$\log_{10} x - 2\log_{10} y = -1$

19 Find the exact value of x such that $\log_x 8 = 6$.

20 Solve the equation $3^{2x} = 2e^x$, giving your answer in terms of natural logarithms.

21 Solve the equation $5^{2x+1} = 7^{x-3}$.

22 Find the solution of the equation $12^{2x} = 4 \times 3^{x+1}$ in the form $\frac{\log p}{\log q}$, where p and q are positive integers.

23 The number of cells in a laboratory experiment satisfies the equation $N = 150e^{1.04t}$, where t hours is the time since the start of the experiment.

a What was the initial number of cells?

b How many cells will there be after 3 hours?

c How long will it take for the number of cells to reach 1000?

24 Simplify $e^{1+3\ln x}$.

25 Given that $\log_{10}\left(\frac{100^x}{10^y}\right)$ can be written as $px + qy$, find the value of p and of q.

26 Let $\log_{10} p = 1.5$ and $\log_{10} q = 2.5$

a Find $\log_{10} p^2$.

b Find $\log_{10}\left(\frac{p}{q}\right)$.

c Find $\log_{10}(10q)$.

27 A geometric sequence has first term 15 and common ratio 1.2. One term of the sequence equals 231, correct to the nearest integer. Which term is it?

28 An infinite geometric series is given by

$(2 - 3x) + (2 - 3x)^2 + (2 - 3x)^3 + \ldots$

a Find the range of values of x for which the series converges.

b Given that the sum of the series is $\frac{1}{2}$, find the value of x.

c Show that the sum of the series cannot equal $-\frac{2}{3}$.

29 The sum to infinity of a geometric series is three times larger than the first term. Find the common ratio of the series.

30 The amount of carbon dioxide absorbed by the Amazon rainforest (C billion tonnes) each year is shown below

Year	2005	2008	2011	2015	2018
C	3.0	2.7	2.5	2.2	2.0

Consider the relationship between the variable .., the number of years after 2004 (so that for example in 2005, $y = 1$) and C.

a Using the natural logarithm find the value of Pearson's product-moment correlation coefficient for
i the log–log graph
ii the semi-log graph.

b Hence suggest a form for the relationship between R and P, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

d Estimate the total amount of carbon dioxide that will be absorbed by the Amazon rain forest after the year 2004, assuming that the trends observed continue.

31 The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

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32 Solve the equation $8^{x-1} = 6^{3x}$. Express your answer in terms of $\ln 2$ and $\ln 3$.

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33 Solve the equation $3^{x+1} = 3^x + 18$.

34 The Arrhenius equation suggests that the rate constant, k seconds^{-1} of an equation depends on the temperature, T in °K according to the model

$$k = Ae^{-\frac{E_a}{RT}}$$

where E is the activation energy (in kJ mol^{-1} per mole) and R is the ideal gas constant, 8.3×10^{-3} kJ mol^{-1} K^{-1}.

a Linearize this equation.

T	280	290	300	310	320
K	4.9×10^{-10}	2.1×10^{-9}	8.5×10^{-9}	3.1×10^{-8}	1×10^{-7}

b Use linear regression to estimate the value of the activation energy.

35 Write in the form $k \ln x$, where k is an integer:

$$\ln x + \ln x^2 + \ln x^3 + \ldots + \ln x^{20}$$

36 Find the exact value of

$$\log_3\left(\frac{1}{3}\right) + \log_3\left(\frac{3}{5}\right) + \log_3\left(\frac{5}{7}\right) + \ldots + \log_3\left(\frac{79}{81}\right)$$

37 Evaluate

$$\sum_{r=0}^{\infty} \frac{3^n + 4^n}{5^n}$$

38 **a** Explain why the geometric series $e^{-x} + e^{-2x} + e^{-3x} + \ldots$ converges for all positive values of x.

b Find an expression for the sum to infinity of the series.

c Given that the sum to infinity of the series is 2, find the exact value of x.

39 A geometric series has sum to infinity 27, and the sum from (and including) the fourth term to infinity is 1. Find the common ratio of the series.

40 Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 6$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

a **i** Show that $\frac{v_{n+1}}{v_n}$ is a constant.

ii Write down the first term of the sequence $\{v_n\}$.

iii Write down a formula for v_n in terms of a, d and n.

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

b **i** Find S_n, in terms of a, d and n.

ii Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

iii Write down S_∞ in terms of a and d.

iv Given that $S_\infty = 2^{a+1}$ find the value of d.

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

c Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q.

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2 Vectors

Sample pages not final

ESSENTIAL UNDERSTANDINGS

- Geometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

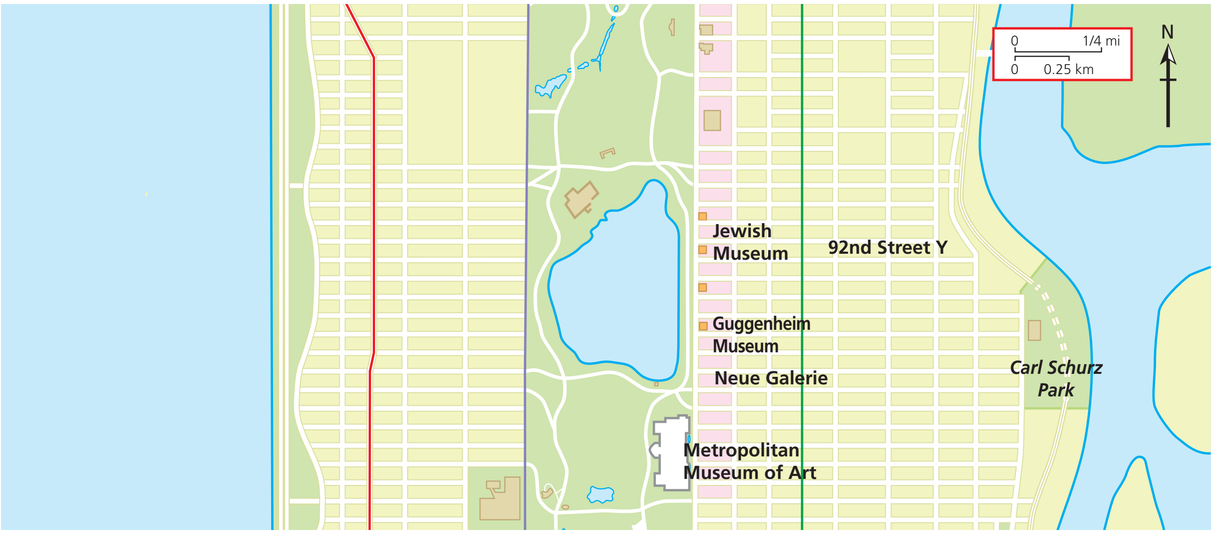
- about the concept of a vector and a scalar
- about different ways of representing vectors and how to add, subtract and multiply vectors by a scalar
- about the resultant of two or more vectors
- how to identify when vectors are parallel
- how to find the magnitude of a vector
- how to find a unit vector in a given direction
- about position vectors and displacement vectors
- how to find the vector equation of a line in two and three dimensions and how to convert to parametric form
- how to determine whether two lines intersect and find the point of intersection
- how to model linear motion with constant velocity in two and three dimensions
- how to use the scalar product to find the angle between two vectors
- how to identify when two vectors are perpendicular
- how to find the angle between two lines
- how to use the vector product to find perpendicular directions and areas
- how to find components of vectors in given directions.

CONCEPTS

The following concepts will be addressed in this chapter:

- The properties of shapes are highly dependent on the dimension they occupy in **space**
- Vectors allow us to **quantify** position, **change** of position (movement) and force in two- and three-dimensional **space**.

Figure 2.1 What information do you need to get from one place to another?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

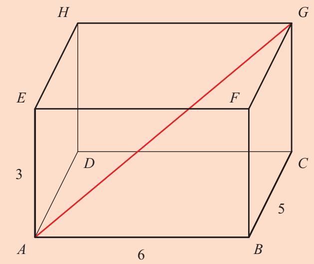
1 In the cuboid below, find the length AG.

2 Find the equation of straight line through the points $(4, 3)$ and $(-1, 5)$.

3 Four points have coordinates $A(3, 2)$, $B(-1, 5)$, $C(1, 6)$ and $D(9, k)$. Find the value of k for which AB and CD are parallel.

4 Solve the simultaneous equations

$$\begin{cases} 3x - 2y = 14 \\ 4x + 5y = 11 \end{cases}$$



You have probably met the distinction between scalar and vector quantities in physics. Scalar quantities, such as mass or time, can be described using a single number. Vector quantities need more than one piece of information to describe them. For example, velocity is described by its direction and magnitude (speed).

In mathematics we use vectors to describe positions of points and displacements between them. Some of this chapter is concerned with using vectors to solve geometrical problems, but there is also a focus on using vectors to describe the motion of object – an area of applied mathematics called kinematics.

Vector equations can describe lines in two- and three-dimensional space. Vector methods enable us to use calculations to determine properties of shapes, such as angles, lengths and areas, in situation which may be difficult to visualize and solve geometrically.

Starter Activity

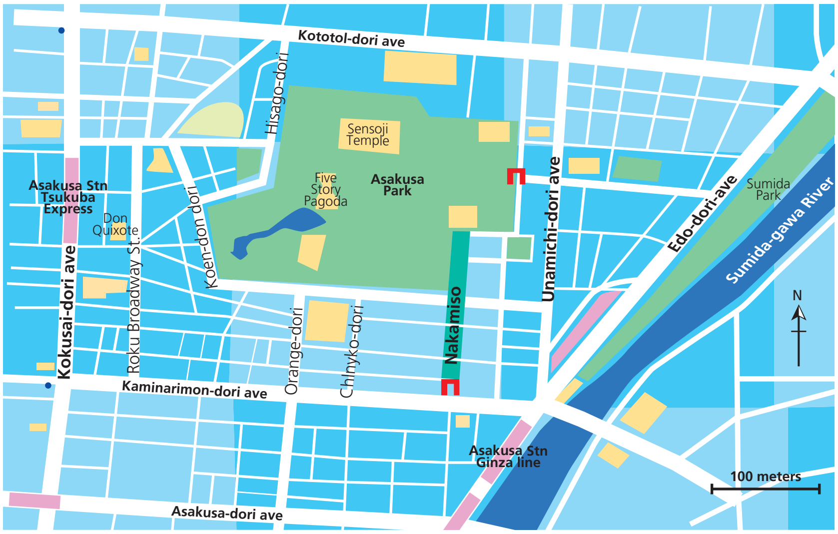
Look at the maps in Figure 2.1. In small groups discuss how you would best give directions from getting from one marked location to another (for example, from the Metropolitan Museum of Art to the 92nd Street Y).

Now look at this problem:

Find the size of the angle between two diagonals of a cube.

LEARNER PROFILE – to follow

Text to be inserted at proof stage.



2A Introduction to vectors

You are the Researcher

It turns out that the given definition of scalars and vectors is slightly simplified. A more formal definition is based in an area of mathematics called tensor analysis, which looks at how quantities change when the frame of reference is rotated.

Concept of a vector and a scalar

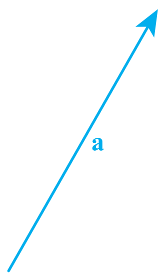
A **vector** is a quantity that has both magnitude and direction, for example force or displacement. This can be represented in several different ways, either graphically or using numbers. The most useful representation depends on the precise application, but you will often need to switch between different representations within the same problem.

A **scalar** is a quantity that has only magnitude but no direction, for example time or energy.

Representation of vectors

A vector is labelled using either a bold lower case letter, for example **a**, or an underlined lower case letter $\underline{a}$.

The simplest way to represent a vector is as a directed line segment, with the arrow showing the direction and the length representing the magnitude, as shown in the margin.

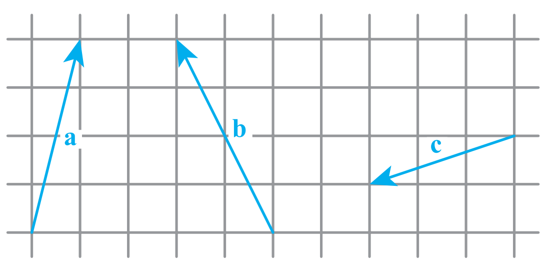


You will see that some problems can be solved using this diagrammatic representation, but sometimes you will also want to do numerical calculations. It that case, it may be useful to represent a vector using its **components**.

In two dimensions you can represent any vector by two numbers. We select two directions, which we will call 'horizontal' and 'vertical'. Then the components of a vector are given by the number of units in the two directions required to get from the 'tail' to the 'head' of the arrow. The components are written as a **column vector**; for example, $\begin{pmatrix}3\\2\end{pmatrix}$ means 3 units to the right and 2 units up.

WORKED EXAMPLE 2.1

Write the following as column vectors (each grid space represents one unit).



The line labelled **a** goes 1 unit to the right and 4 units up ······ $\mathbf{a} = \begin{pmatrix}1\\4\end{pmatrix}$

The line labelled **b** goes two units to the left and four units up ······ $\mathbf{b} = \begin{pmatrix}-2\\4\end{pmatrix}$

The line labelled **c** goes three units to the left and one unit down ······ $\mathbf{c} = \begin{pmatrix}-3\\-1\end{pmatrix}$

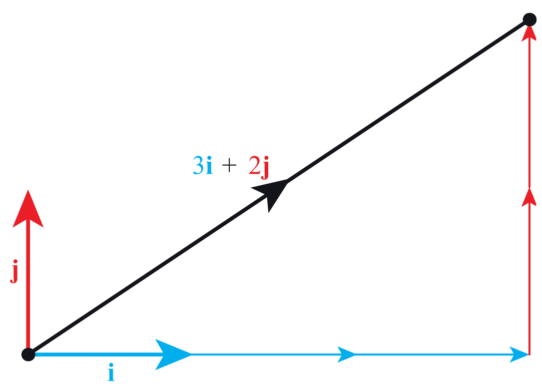
Tip

We often denote a vector by a single letter, just like variables in algebra. In printed text the letter is usually bold; when writing you should underline it (for example, $\underline{a}$) or use an arrow (for example, $\vec{a}$) to distinguish between vectors and scalars.

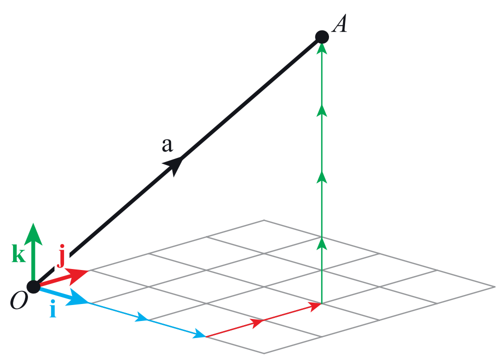
Base vectors

Another way to write a vector in components is to use **base vectors**, denoted **i** and **j** in two dimensions. These are vectors of length 1 in the directions 'right' and 'up'.

For example, the vector $\begin{pmatrix}3\\2\end{pmatrix}$ can be written $3\mathbf{i} + 2\mathbf{j}$.



This approach can be extended to three dimensions. We need three base vectors, called **i, j, k**, all perpendicular to each other.



$$\mathbf{a} = \begin{pmatrix}3\\2\\4\end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Tip

The base vectors written as column vectors are:

$$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Tip

You must be familiar with both base vector and column vector notation, as both are frequently used. When you write your answers, you can use whichever notation you prefer.

WORKED EXAMPLE 2.2

a Write the following using base vectors.

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$$

b Write the following as column vectors with three components.

$\mathbf{d} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{e} = 2\mathbf{j} - \mathbf{k}, \quad \mathbf{f} = 3\mathbf{k} - \mathbf{i}$

a

The components are the coefficients of **i** and **j**

$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$

The components are coefficients of **i**, **j** and **k**

$\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Notice that the **j**-component is zero

$\mathbf{c} = 4\mathbf{i} - 2\mathbf{k}$

b

The coefficients are the components of the vectors

$$\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Notice that the **i**-component is missing

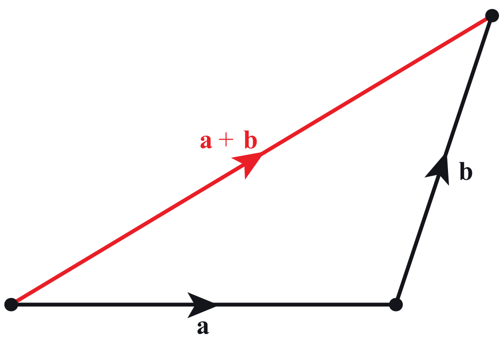
$$\mathbf{e} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Be careful – the components are not in the correct order!

$$\mathbf{f} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

Addition and subtraction of vectors

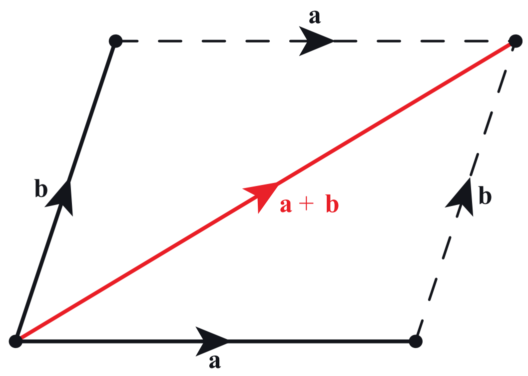
On a diagram, vectors are added by joining the starting point of the second vector to the end point of the first. In component form, you just add the corresponding components.



$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

The vector that results from the sum of two or more vectors is known as the **resultant vector**. So, in the above diagram, the resultant of the vectors $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$.

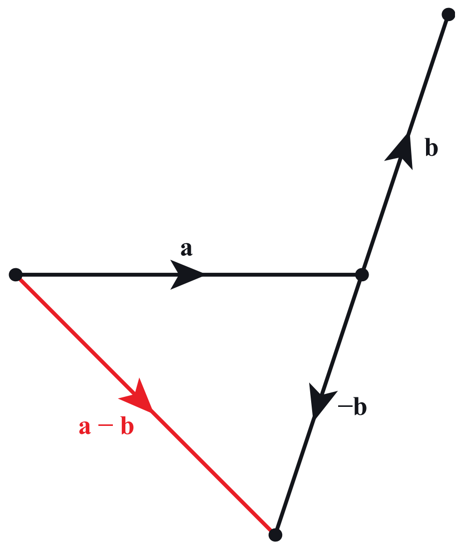
Another way of visualizing vector addition is as a diagonal of the parallelogram formed by the two vectors.



Tip

Equal vectors have the same magnitude and direction; they don't need to start or end at the same point.

To subtract vectors, reverse the direction of the second vector and add it to the first. Notice that subtracting a vector is the same as adding its negative. For example, if $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.



$$\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Tip

Subtracting a vector from itself gives the zero vector: $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

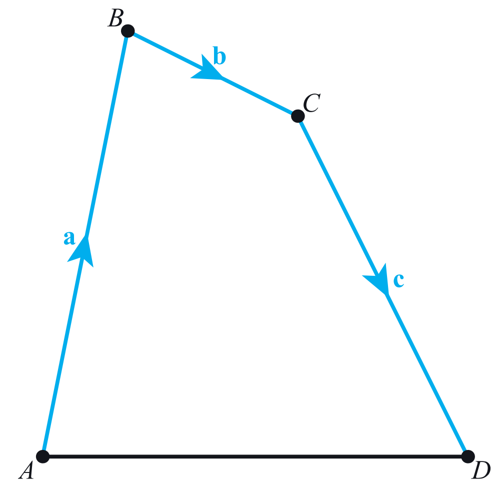
Another way of writing vectors is to label the end-points with capital letters. For example, $\overrightarrow{AB}$ is the vector in the direction from A to B, with magnitude equal to the length AB.

WORKED EXAMPLE 2.3

Express the following in terms of vectors **a**, **b** and **c**.

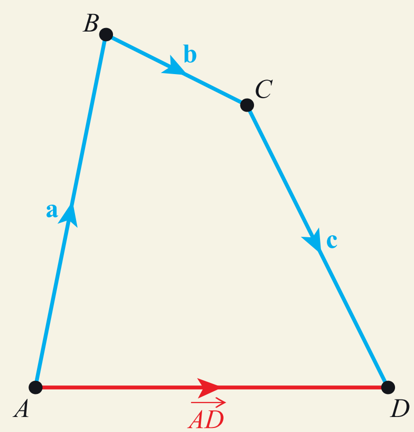
a $\overrightarrow{AD}$

b $\overrightarrow{DB}$



The vectors **a**, **b** and **c** are joined 'head to tail', starting at A and finishing at D

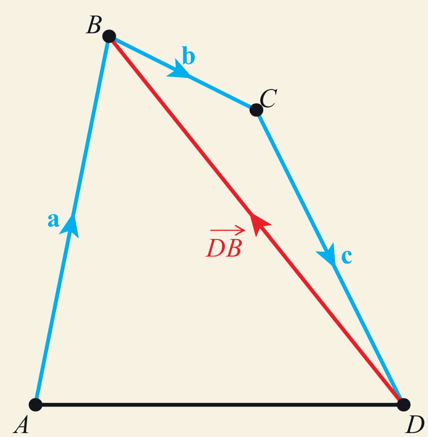
a



$\overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

You can see that $\mathbf{b} + \mathbf{c} = \overrightarrow{BD}$, and $\overrightarrow{DB}$ is the negative of this

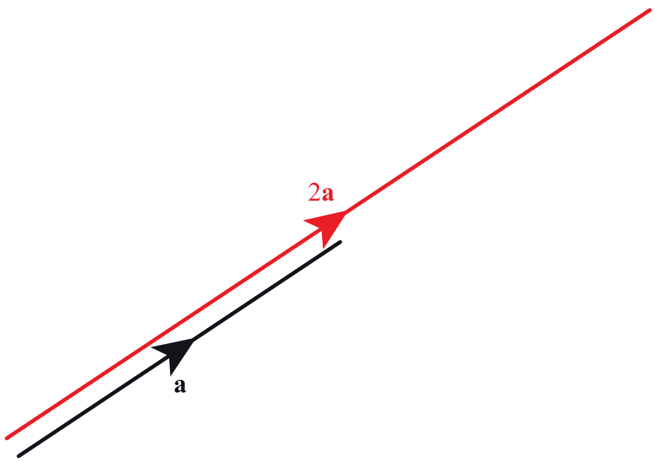
b



$$\begin{aligned}\overrightarrow{DB} &= -\overrightarrow{BD}\\ &= -(\mathbf{b} + \mathbf{c})\\ &= -\mathbf{b} - \mathbf{c}\end{aligned}$$

Scalar multiplication and parallel vectors

Multiplying by a scalar changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.



$$2\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$

Tip

Multiplying by a negative scalar reverses the direction.

WORKED EXAMPLE 2.4

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$, find $2\mathbf{a} - 3\mathbf{b}$.

$$2\mathbf{a} - 3\mathbf{b} = 2\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

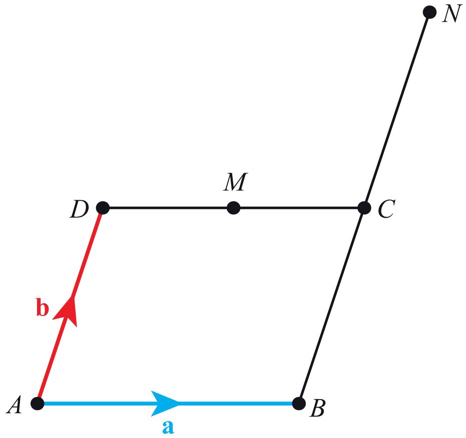
Multiply the scalar by each element of the relevant vector then subtract corresponding components of the vectors

$$= \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} - \begin{pmatrix} -9 \\ 12 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ -8 \\ 8 \end{pmatrix}$$

WORKED EXAMPLE 2.5

The diagram shows a parallelogram $ABCD$. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of CD and N is the point on BC such that $CN = BC$.



a Express vectors $\overrightarrow{CM}$ and $\overrightarrow{BN}$ in terms of **a** and **b**.

b Express $\overrightarrow{MN}$ in terms of **a** and **b**.

$\overrightarrow{CM}$ has the same direction as $\overrightarrow{BA}$, but half the length ⋯ **a** $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA}$

$\overrightarrow{BA} = -\overrightarrow{AB}$ ⋯ $= -\frac{1}{2}\mathbf{a}$

$\overrightarrow{BN}$ is twice $\overrightarrow{BC}$… ⋯ $\overrightarrow{BN} = 2\overrightarrow{BC}$

… which is the same as $\overrightarrow{AD}$ ⋯ $= 2\mathbf{b}$

b $\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$

We can think of $\overrightarrow{MN}$ as describing a way of getting from C to M moving only along the directions of **a** and **b**. This can be achieved by going from M to C and then from C to N ⋯ $= \frac{1}{2}\mathbf{a} + \mathbf{b}$

Two vectors are parallel if they have the same (or opposite) direction. This means that one is a scalar multiple of the other.

Tip

The scalar can be positive or negative.

KEY POINT 2.1

If vectors **a** and **b** are parallel, we can write $\mathbf{b} = t\mathbf{a}$ for some scalar t.

CONCEPTS – QUANTITY

We can do much more powerful mathematics with **quantities** than with drawings. Although the concept of parallel lines is a geometric concept, it is useful to quantify it in order to use it in calculations and equations. Key Point 8.1 gives us an equation to express the geometric statement 'two lines are parallel'.

WORKED EXAMPLE 2.6

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$, find the values of p and q such that $\mathbf{c}$ is parallel to $\mathbf{a}$.

Write $\mathbf{c} = t\mathbf{a}$ for some scalar t.
Then,

If two vectors are parallel, we can write $\boldsymbol{v}_1 = t\boldsymbol{v}_2$

$$\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$$

If two vectors are equal, then all their components are equal

$$\begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$p = -4,\ q = -14$

Magnitude of a vector and unit vectors

The magnitude of a vector can be found from its components, using Pythagoras' theorem. The symbol for the magnitude is the same as the symbol for absolute value (modulus).

KEY POINT 2.2

The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

In some applications it is useful to make vectors have length 1. These are called **unit vectors**. The base vectors **i**, **j**, **k** are examples of unit vectors, but you can create a unit vector in any direction. You can take any vector in that direction and divide it by its magnitude; this will keep the direction the same but change the magnitude to 1.

KEY POINT 2.3

The unit vector in the same direction as vector $\mathbf{a}$ is $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

Tip

Note that there are two possible answers to part **b**, as you could multiply a by $-\frac{1}{3}$ instead of $\frac{1}{3}$.

WORKED EXAMPLE 2.7

a Find the magnitude of the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a unit vector parallel to $\mathbf{a}$.

Use Pythagoras

a $|\mathbf{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

Divide by the magnitude of **a** to create a vector of length 1

b Unit vector is $\frac{1}{3}\mathbf{a} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

Position and displacement vectors

In Section C you will find out how to use vectors in kinematics.

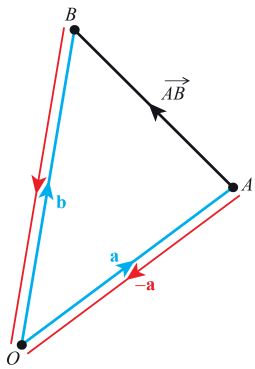
Vectors can be used to represent many different quantities, such as force, velocity or acceleration. They always obey the same algebraic rules you learnt in the previous section. One of the most common applications of vectors in pure mathematics is to represent positions of points in space.

You already know how to use coordinates to represent the position of a point, measured along the coordinate axes from the origin O. The vector from the origin to a point A is called the **position vector** of A. The base vectors **i**, **j**, and **k** are the unit vectors in the direction of x, y and z axes, respectively.

KEY POINT 2.4

- The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, where O is the origin.
- The components of $\mathbf{a}$ are the coordinates of A.

Position vectors describe positions of points relative to the origin, but you sometimes want to know the position of one point relative to another. This is described by a **displacement vector**.



KEY POINT 2.5

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then the displacement vector from A to B is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

Tip

You can think of the equation $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ as saying: 'to get from A to B, go from A to O and then from O to B'.

WORKED EXAMPLE 2.8

Points A and B have coordinates $(3, -1, 2)$ and $(5, 0, 3)$. Find the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point

$$\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

The displacement is the difference between the position vectors (end − start)

$$= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The distance between two points is equal to the magnitude of the displacement vector.

Tip

The displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have equal magnitude but opposite direction.

KEY POINT 2.6

The distance between the points A and B with position vectors **a** and **b** is

$$AB = |\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$$

WORKED EXAMPLE 2.9

Points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the exact distance AB.

First find the displacement vector

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$
$$= (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (3\mathbf{i} - \mathbf{j} - 4\mathbf{k})$$
$$= -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

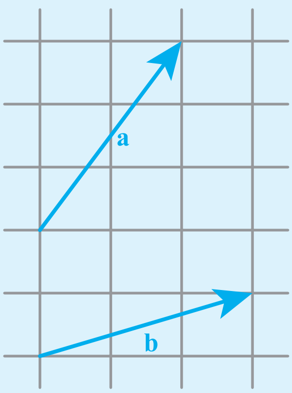
The distance is the magnitude of the displacement vector

$$|\overrightarrow{AB}| = \sqrt{4+1+9}$$
$$= \sqrt{14}$$

Exercise 2A

For questions 1 to 4, use the method demonstrated in Worked Example 2.1 to write vectors **a** and **b** as column vectors.

1



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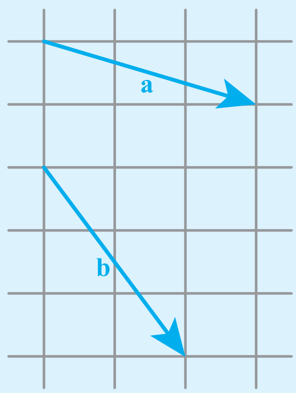

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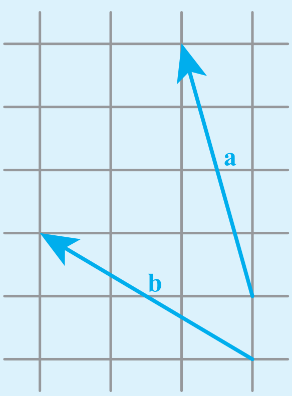

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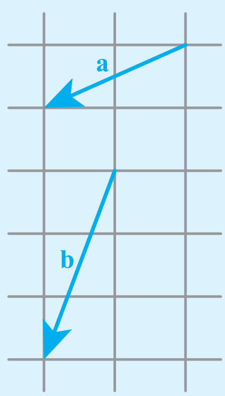


For questions 5 to 7, use the method demonstrated in Worked Example 2.2 to write the following using base vectors **i**, **j**, **k**.

5 **a** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

6 **a** $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

7 **a** $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$

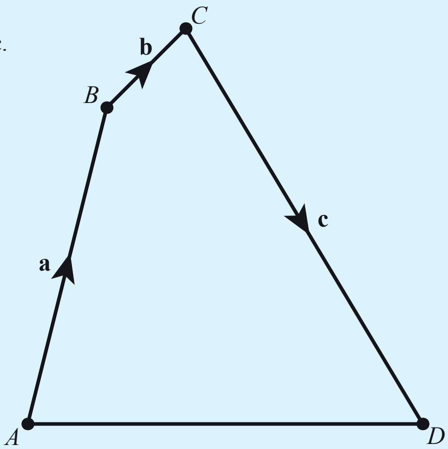
For questions 8 to 10, use the method demonstrated in Worked Example 2.2 to write the following as three-dimensional column vectors.

8 **a** $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ **b** $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

9 **a** $\mathbf{i} + 3\mathbf{k}$ **b** $2\mathbf{j} - \mathbf{k}$

10 **a** $4\mathbf{j} - \mathbf{i} - 2\mathbf{k}$ **b** $\mathbf{k} - 3\mathbf{i}$

In questions 11 and 12, use the method demonstrated in Worked Example 2.3 to express the following vectors in terms of **a**, **b** and **c**.



11 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{BD}$

12 **a** $\overrightarrow{DA}$ **b** $\overrightarrow{DB}$

For questions 13 to 15, you are given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$.

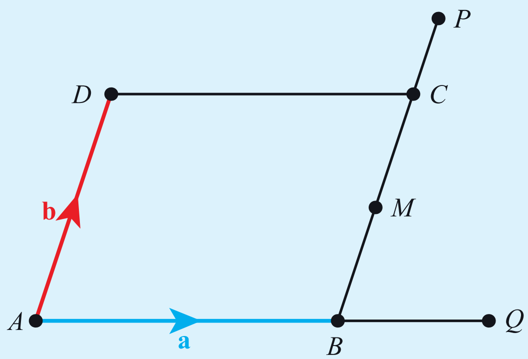
Use the method demonstrated in Worked Example 2.4 to write the following as column vectors.

13 **a** $\mathbf{a} - \mathbf{b}$ **b** $\mathbf{b} - \mathbf{a}$

14 **a** $3\mathbf{a} + 2\mathbf{b}$ **b** $2\mathbf{a} + 5\mathbf{b}$

15 **a** $3\mathbf{b} - \mathbf{a}$ **b** $\mathbf{b} - 2\mathbf{a}$

For questions 16 to 18, *ABCD* is a parallelogram, with $\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{a}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{a}$. *M* is the midpoint of *BC*, *Q* is the point on the extended line *AB* such that $BQ = \frac{1}{2}AB$ and *P* is the point on the extended line *BC* such that $CP = \frac{1}{3}BC$, as shown on the diagram.



Use the method demonstrated in Worked Example 2.5 to write the following vectors in terms of **a** and **b**.

16 **a** $\overrightarrow{AP}$ **b** $\overrightarrow{AM}$

17 **a** $\overrightarrow{QD}$ **b** $\overrightarrow{MQ}$

18 **a** $\overrightarrow{DQ}$ **b** $\overrightarrow{PQ}$

For questions 19 to 21, use the method demonstrated in Worked Example 2.6 to find the value of p and q such that the vectors **a** and **b** are parallel.

19 a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ p \\ q \end{pmatrix}$

20 a $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -9 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$

21 a $\mathbf{a} = \begin{pmatrix} 2 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 2 \\ 3 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -3 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 15 \\ 2 \end{pmatrix}$

22 a $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + 6\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + q\mathbf{j} + 6\mathbf{k}$

23 a $\mathbf{a} = 2\mathbf{i} + p\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 4\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = p\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + q\mathbf{k}$

For questions 24 to 27, use the method demonstrated in Worked Example 2.7 to find the unit vector in the same direction as vector **a**.

24 a $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

25 a $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

26 a $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

b $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

27 a $\mathbf{a} = \mathbf{i} - 4\mathbf{j}$

b $\mathbf{a} = 2\mathbf{j} - 3\mathbf{k}$

For questions 28 to 30, points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Use the method demonstrated in Worked Example 2.8 to find the given displacement vectors.

28 a $\overrightarrow{AB}$

b $\overrightarrow{AC}$

29 a $\overrightarrow{CB}$

b $\overrightarrow{CA}$

30 a $\overrightarrow{BA}$

b $\overrightarrow{BC}$

For questions 4 to 6, use the method demonstrated in Worked Example 2.9 to find the exact distance between the points A and B with the given position vectors.

31 a $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

b $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

32 a $\mathbf{a} = \mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$

b $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$

33 a $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

34 Given that $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ is the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 12 \\ y \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -5 \\ 2 \\ z \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} x \\ 9 \\ 4 \end{pmatrix}$, find the value of x, y and z.

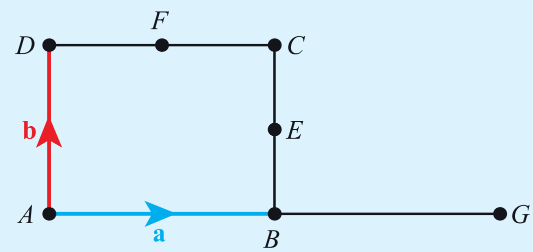
35 The diagram shows a rectangle $ABCD$. E is the midpoint of BC, F is the midpoint BC and G is the point on the extension of the side AB such that $BG = AB$.

Define vectors $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{a} = \overrightarrow{AD}$. Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

a $\overrightarrow{AE}$

b $\overrightarrow{EF}$

c $\overrightarrow{DG}$



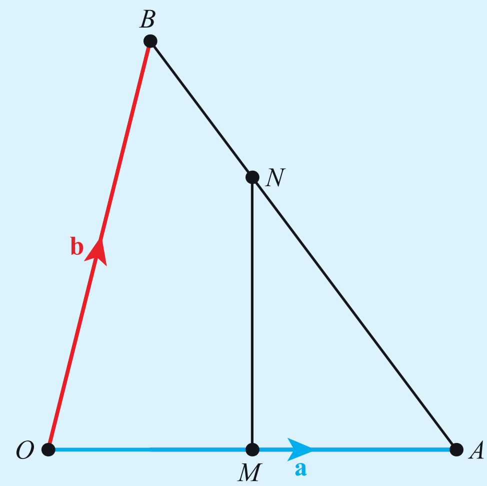
36 In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA and N is the point on AB such that $BN = \frac{1}{3}BA$.

Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

a $\overrightarrow{BA}$

b $\overrightarrow{ON}$

c $\overrightarrow{MN}$



37 Given the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, find

a $3\mathbf{a} - \mathbf{c} + 5\mathbf{b}$

b $|\mathbf{b} - 2\mathbf{a}|$.

38 Find the possible values of the constant k such that the vector $\begin{pmatrix} 3k \\ -k \\ k \end{pmatrix}$ has magnitude 22.

39 The vector $2\mathbf{i} + 3t\mathbf{j} + (t - 1)\mathbf{k}$ has magnitude 3. Find the possible values of t.

40 An object is acted on by three forces:

$$\mathbf{F}_1 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}\text{N},\ \mathbf{F}_2 = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}\text{N},\ \mathbf{F}_3 = \begin{pmatrix} 0 \\ -4 \\ 7 \end{pmatrix}\text{N}$$

Find the magnitude of the resultant force acting on the object.

41 Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$ find vector $\boldsymbol{x}$ such that $3\mathbf{a} + 4\boldsymbol{x} = \mathbf{b}$.

42 Given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{k}$, find the value of the scalar t such that $\mathbf{a} + t\mathbf{b} = \mathbf{c}$.

43 a Find a unit vector parallel to $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$.

b Find a vector of magnitude 10 in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

44 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ find the value of the scalar p such that $\mathbf{a} + p\mathbf{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

45 Given that $\boldsymbol{x} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\boldsymbol{y} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ find the value of the scalar λ such that $\lambda\boldsymbol{x} = \boldsymbol{y}$ is parallel to vector $\mathbf{j}$.

46 Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

47 Let $\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$.

48 Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ where O is the origin. Find the possible values of t such that $AB = 3$.

49 Points P and Q have position vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

a Find the position vector of the midpoint M of PQ.

b Point R lies on the line PQ such that $QR = QM$. Find the coordinates of R.

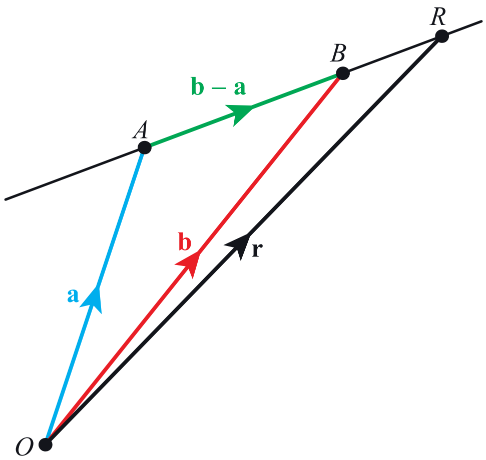
50 Find the smallest possible magnitude of the vector $(2t)\mathbf{i} + (t+3)\mathbf{j} - (2t+1)\mathbf{k}$, where t is a real constant.

2B Vector equation of a line

Consider a straight line through points A and B, with position vectors **a** and **b**. For any other point R on the line, the vector $\overrightarrow{AR}$ is in the same direction as $\overrightarrow{AB}$, so you can write $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ for some scalar λ. Using position vectors, this equation becomes:

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$

This can be rearranged to express the position vector **r** in terms of **a**, **b** and λ.



KEY POINT 2.7

The equation of the line through points with position vectors **a** and **b** is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

Tip

You can write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to find the coordinates of a point on the line.

Different values of λ give position vectors of different points on the line. For example, you can check $\lambda = 0$ gives point A, $\lambda = 1$ gives point B, and $\lambda = 0.5$ gives $\mathbf{r} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, which is the midpoint of AB.

WORKED EXAMPLE 2.10

a Find the equation of the straight line through the point $A(2, -1, 3)$ and $B(1, 1, 5)$.

b Determine whether the point $C(1.5, 0, 4.5)$ lies on this line.

a

First find the vector $\mathbf{b}-\mathbf{a}$

$$\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\1\\5\end{pmatrix}-\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

The equation of line is:

Use $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$

$$\mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

b

Find λ such that

Is there a value of λ which gives $\mathbf{r}=\mathbf{c}$?

$$\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}1.5\\0\\4.5\end{pmatrix}$$

This represents three equations, one for each component

$$\begin{cases}2-\lambda=1.5 & (1)\\ -1+2\lambda=0 & (2)\\ 3+2\lambda=4.5 & (3)\end{cases}$$

The same value of λ needs to satisfy all three equations. Find λ from the first equation and check in the other two

(1): $\lambda=0.5$

(2): $-1+2(0.5)=0$

(3): $3+2(0.5)=4\neq 4.5$

There is no value of λ which satisfies all three equations

Point C does not lie on the line.

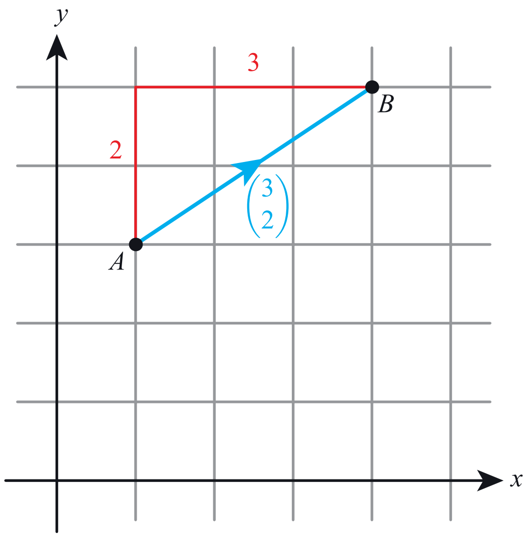
You will learn more about recognizing which equations describe the same line in the next section.

Tip

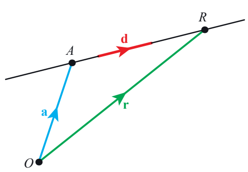
You could have used point B instead of A to write the equation in Worked Example 8.14 as $\mathbf{r}=\begin{pmatrix}1\\1\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$. Or you could use vector $\overrightarrow{BA}$ instead of $\overrightarrow{AB}$ to get $\mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$.
All of those equations represent the same line, but the value of λ for a given point on the line is different for different equations. For example, the point $(0, 3, 7)$ corresponds to $\lambda=1$ in the first equation, and $\lambda=-2$ in the second equation.

The vector equation of a line takes the same form in two dimensions. For example, the equation of the line through the points $(1, 3)$ and $(4, 5)$ can be written as $\mathbf{r}=\begin{pmatrix}1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\end{pmatrix}$.

If you draw a diagram, you can see how the vector $\mathbf{b} - \mathbf{a}$ is related to the gradient of the line:



The vector $\mathbf{b} - \mathbf{a}$ is a **direction vector** of the line. You can find the equation of a line if you know only one point and a direction vector.



KEY POINT 2.8

A vector equation of the line with direction vector $\mathbf{d}$ passing through the point $\mathbf{a}$ is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$.

WORKED EXAMPLE 2.11

Write down a vector equation of the line with direction vector $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ passing through the point $(-3, 3, 5)$.

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{d}$ is a direction vector

$$\mathbf{r} = \begin{pmatrix}-3\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\2\end{pmatrix}$$

Tip

You can use any multiple of $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ as a direction vector.

Parametric form of the equation of a line

You can rewrite the vector equation of a line as three separate equations for x, y and z.

To do this, just remember that $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$.

KEY POINT 2.9

The **parametric form** of the equation of a line is found by expressing x, y and z in terms of λ.

WORKED EXAMPLE 2.12

Write the parametric form of the equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$.

Write r as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$$

Write a separate equation for each component

$$\Rightarrow \begin{cases} x = 3 - 4\lambda \\ y = -2 \\ z = 5\lambda \end{cases}$$

Tip

When working with two different lines, use two different letters (such as λ and μ) for the parameters.

Finding the point of intersection of two lines

Suppose two lines have vector equations $\mathbf{r}_1 = \mathbf{a} + \lambda\mathbf{d}_1$ and $\mathbf{r}_2 = \mathbf{b} + \mu\mathbf{d}_2$. If they intersect, then there is a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\mathbf{r}$, this means that we need to find the values of λ and μ which make $\mathbf{r}_1 = \mathbf{r}_2$.

WORKED EXAMPLE 2.13

Find the coordinates of the point of intersection of the following pair of lines.

$$\mathbf{r} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

You need to make $\mathbf{r}_1 = \mathbf{r}_2$

$$\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

Write three separate equations, one for each component

$$\Rightarrow \begin{cases} 0 + \lambda = 1 + 4\mu \\ -4 + 2\lambda = 3 - 2\mu \\ 1 + \lambda = 5 - 2\mu \end{cases}$$

$$\Rightarrow \begin{cases} \lambda - 4\mu = 1 & (1) \\ 2\lambda + 2\mu = 7 & (2) \\ \lambda + 2\mu = 4 & (3) \end{cases}$$

$$(3) - (1) \Rightarrow 6\mu = 3$$

Pick two equations to solve, then check the answers in the third. This case, subtracting (1) from (3) eliminates λ

$$\mu = \frac{1}{2}, \lambda = 3$$

The values of λ and μ you have found must also satisfy equation (2)

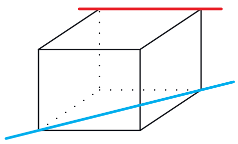
$$(2): \ 2 \times 3 + 2 \times \frac{1}{2} = 7$$

The lines intersect.

The position of the intersection point is given by the vector $\mathbf{r}_1$ (or $\mathbf{r}_2$ – they should be the same – you should always check this)

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The lines intersect at the point (3,2 4).



In two dimensions, two distinct lines either intersect or are parallel. In three dimensions it is possible for the lines to neither intersect nor be parallel.

CONCEPTS – SPACE

Some properties of objects depend on the dimension they occupy in **space**. One of the most interesting examples of this is diffusion, which is very important in physics and biology. If a large number of particles move randomly (performing a so-called random walk) in three dimensions, on average they will keep moving away from the starting point. This is, however, not the case in one or two dimensions where, on average, the particles will return to the starting point.

If two lines do not intersect, it is impossible to find the values of λ and μ which solve all three equations.

WORKED EXAMPLE 2.14

Show that the lines with equations $\mathbf{r} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ are skew.

Try to make $\mathbf{r}_1 = \mathbf{r}_2$ and then show that this is not possible

$$\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} t - 2\lambda = 6 & (1) \\ t + 3\lambda = -2 & (2) \\ 4t - 2\lambda = -2 & (3) \end{cases}$$

Find t and λ from the first two equations

(1) and (2) $\Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$

The values found must also satisfy the third equation

(3): $4\left(\frac{14}{5}\right) - 2\left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$

This tells you that it is impossible to find t and λ to make $\mathbf{r}_1 = \mathbf{r}_2$

The two lines do not intersect.

The lines are not parallel:

The two lines don't intersect, so they could be parallel or skew. Parallel lines have parallel direction vectors

$$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \neq k\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

Hence, the two lines are skew.

Exercise 2B

For questions 1 to 3, use the method demonstrated in Worked Example 2.10 to find the equation of the line through A and B, and determine whether point C lies on the line.

1 **a** $A(2, 1, 5)$, $B(1, 3, 7)$, $C(0, 5, 9)$

b $A(-1, 0, 3)$, $B(3, 1, 8)$, $C(-5, -1, 3)$

2 **a** $A(4, 0, 3)$, $B(8, 0, 6)$, $C(0, 0, 2)$

b $A(-1, 5, 1)$, $B(-1, 5, 8)$, $C(-1, 3, 8)$

3 **a** $A(4, 1)$, $B(1, 2)$, $C(5, -2)$

b $A(2, 7)$, $B(4, -2)$, $C(1, 11.5)$

For questions 4 to 6, use the method demonstrated in Worked Example 2.11 to write down a vector equation of the line with the give direction vector passing through the given point.

4 **a** Point (1, 0, 5), direction $\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

b Point (−1, 1, 5), direction $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

5 **a** Direction $\mathbf{i} - 3\mathbf{k}$, point (0, 2, 3)

b Direction $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, point (4, −3, 0)

6 **a** Direction $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, point (4, −1)

b Direction $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, point (4, 1)

For questions 7 to 9, use the method demonstrated in Worked Example 2.12 to write the parametric form of the equation of the line with the given vector equation.

7 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

8 **a** $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j}) + \lambda(\mathbf{i} - 4\mathbf{k})$

b $\mathbf{r} = (3\mathbf{j} - \mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j})$

9 **a** $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

For questions 10 to 12, use the method demonstrated in Worked Example 2.13 to find the coordinates of the point of intersection of the two lines.

10 **a** $\mathbf{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

11 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

12 **a** $\mathbf{r} = (3\mathbf{i} + \mathbf{j}) + \lambda(2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$

b $\mathbf{r} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (8\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

For questions 13 and 14, use the method demonstrated in Worked Example 2.14 to show that the two lines do not intersect.

13 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ -11 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

14 **a** $\mathbf{r} = (\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (3\mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} + \mathbf{k})$

b $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ and $\mathbf{r} = (3\mathbf{i} + 2\mathbf{k}) + \mu(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$

15 **a** Find a vector equation of the line passing through the points (3, –1, 5) and (–1, 1, 2).

b Determine whether the point (0, 1, 5) lies on the line.

16 Find the parametric form of the equation of the line passing through the point (–1, 1, 2) parallel to the line with vector equation $\mathbf{r} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{i}) + \lambda(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$.

17 Determine whether the point $A(3, -2, 2)$ lies on the line with equation $x = 2\lambda - 1,\ y = 4 - 3\lambda,\ z = 1.5\lambda$.

18 **a** Find a vector equation of the line with parametric equation

$x = 3 - 2\lambda,\ y = -1,\ z = \dfrac{4\lambda}{3} - 1$

b Find the value of p so that the point $(2, -1, p)$ lies on the line.

19 A line is given by parametric equations $x = 3 - \lambda,\ y = 4\lambda,\ z = 2 + \lambda$.

The point $(0, p, q)$ lies on the line. Find the values of p and q.

20 Show that these two lines intersect, and find the coordinates of the intersection point:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

21 Determine whether or not these two lines intersect:

$l_1:\ x = -3 - \lambda,\ y = 5 - 2\lambda,\ z = 2 - 4\lambda$

$l_2:\ x = 8 - \mu,\ y = 5 - 3\mu,\ z = 1 + 3\mu$

22 **a** Show that the equations $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $x = 5 + 6\lambda,\ y = 7 + 6\lambda,\ z = 5 + 3\lambda$ represent the same straight line.

b Show that the equation $\mathbf{r} = (4t - 5)\mathbf{i} + (4t - 3)\mathbf{j} + (1 + 2t)\mathbf{k}$ represents a different straight line.

23 **a** Show that the points $A(4, -1, -8)$ and $B(2, 1, -4)$ lie on the line l with equation $r = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

b Find the coordinates of the point C on the line l such that $AB = BC$.

24 **a** Find the vector equation of line l through points $P(7, 1, 2)$ and $Q(3, -1, 5)$.

b Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R.

25 **a** Write down the vector equation of the line l through the point $A(2, 1, 4)$ parallel to the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

b Calculate the magnitude of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

c Find the possible coordinates of point P on l such that $AP = 35$.

26 Points A, B, C and D have coordinates $A(1, 0, 0)$, $B(6, 5, 5)$, $C(8, 3, 3)$, $D(6, 3, 3)$. Find the point of intersection of the lines AB and CD.

27 **a** Find the coordinates of the point where the line with equation $\mathbf{r} = \begin{pmatrix} 6 \\ -1 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$ intersects the y-axis.

b Show that the line does not intersect the z-axis.

28 Find the value of p for which the lines with equations $\mathbf{r} = (\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}) + \lambda(\mathbf{i} + p\mathbf{k})$ intersect. Find the point of intersection in this case.

29 Find the distance of the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

30 Find the shortest distance from the point (–1, 1, 2) to the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

2C Vector applications to kinematics

We can now apply the methods of the previous sections to allow us to model the motion of objects moving with constant velocity.

You have already met the idea of the position vector of an object being the vector that gives the coordinates of that object. As well as knowing where an object is at any given point in time, it is also useful to know which direction it is moving in and how fast it is moving. This information is given by the **velocity vector**.

KEY POINT 2.10

An object with constant velocity **v** is moving parallel to the vector **v** with speed $|\mathbf{v}|$.

WORKED EXAMPLE 2.15

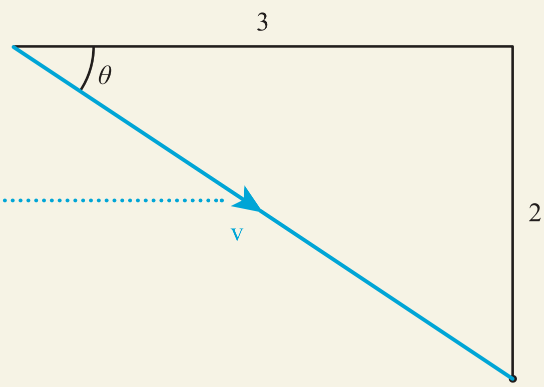
An object is moving with velocity $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ m s^{-1}.

Find

a The object's direction of motion, as a bearing.

b The object's speed.

a



It is helpful to draw a diagram

Find θ using trigonometry $\theta = \tan^{-1}\frac{2}{3} = 33.7°$

And convert to a bearing Bearing $= 90 + 33.7 = 124°$

b

$$\text{Speed} = \left|\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right|$$

Speed $= |\mathbf{v}|$ $= \sqrt{3^2 + (-2)^2}$

$$= 3.61 \text{ m s}^{-1}$$

You saw how to find unit vectors, and vectors of given magnitude in a certain direction in Section 2A.

WORKED EXAMPLE 2.16

Find the velocity vector of an object moving in the direction $\begin{pmatrix}1\\3\\4\end{pmatrix}$ with speed $10\,\text{m s}^{-1}$.

First find a unit vector in the direction of $\begin{pmatrix}1\\3\\4\end{pmatrix}$

$$= \frac{1}{\sqrt{1^2+3^2+4^2}}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

$$= \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

So,

The multiply by 10 so that $|\mathbf{v}| = 10$

$$\mathbf{v} = \frac{10}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}\text{m s}^{-1}$$

If an object is moving with constant velocity it will move in a straight line. The displacement from the starting position after time t will be $t\mathbf{v}$, where $\mathbf{v}$ is the velocity vector. The actual position of the object (relative to the origin) can be found by adding this displacement to the initial position vector.

KEY POINT 2.11

For an object moving with constant velocity $\mathbf{v}$ from the starting position $\mathbf{r}_0$, the position after time t is given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

Notice that the equation in Key Point 2.11 represents a straight line with direction vector $\mathbf{v}$, where t plays the role of the parameter λ. It is indeed the equation of the line along which the object moves.

WORKED EXAMPLE 2.17

An object moves with constant velocity $\mathbf{v} = (2\mathbf{i} - \mathbf{j} + 5\mathbf{k})\,\text{m s}^{-1}$. Its position vector when $t = 0$ is $\mathbf{r}_0 = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\,\mathbf{m}$.

a Write down an equation for the position of the object at time t seconds.

b Find the distance of the object from the origin when $t = 5$ seconds.

Use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

a $\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

b When $t = 5$:

First find the position vector when $t = 5$

$$\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + 5(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$
$$= (14\mathbf{i} - 2\mathbf{j} + 21\mathbf{k})$$

Distance is the magnitude of the displacement vector

$$\text{distance} = |\mathbf{r}| = \sqrt{14^2 + 2^2 + 21^2}$$
$$= 25.3\text{ m}$$

Tip

Whereas before you needed to use different parameters for the different lines, you now have the same parameter, t, for both lines. This means the lines could intersect but the particles do not collide as they are not both at the same point at the same time.

TOOLKIT: Modelling

The velocity of an aeroplane is modelled by the constant vector $(p\mathbf{i} + q\mathbf{j} + 0\mathbf{k})\,\text{km h}^{-1}$.

a Suggest suitable directions for the unit base vectors **i**, **j** and **k**.

b What assumptions have been made in this model? Are those assumptions reasonable?

c Can you suggest other ways of modelling the motion of an aeroplane?

Now that you have the equation of a straight line defining the paths of objects, you can use the methods of Section 2B to determine whether objects meet or if not and how close they get to each other.

WORKED EXAMPLE 2.18

The position vectors (in km) of two model aircraft at time t hours are given by

$$\mathbf{r}_A = \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$$

a Show that they do not collide.

b Find the time at which they are closest together, and their distance apart at this point.

As before for showing that two lines do not intersect, try to make $\mathbf{r}_A = \mathbf{r}_B$ and show that this is not possible

a $$\begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$$

The difference now is that you do not use a different parameter on each side of the equations. If they do intersect, this must happen at the same time, t.

$$\Rightarrow \begin{cases} 2 + t = -1 + 6t & (1) \\ -6 + 4t = -3 - t & (2) \\ 3t = 1 + 2t & (3) \end{cases}$$

Solve each equation for t. The first two give the same value of t.

$$(1): t = \frac{3}{5}$$

$$(2): t = \frac{3}{5}$$

However, the third gives a different value

$$(3): t = 1 \neq \frac{3}{5}$$

There is no time at which all three coordinates are equal

So the hikers never meet.

The distance between A and B is $|\overrightarrow{AB}|$ so first find the displacement vector of A from B

b $\overrightarrow{AB} = \mathbf{r}_B - \mathbf{r}_A$

$$= \begin{pmatrix} -1+6t \\ -3-t \\ 1+2t \end{pmatrix} - \begin{pmatrix} 2+t \\ -6+4t \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} -3+5t \\ 3-5t \\ 1-t \end{pmatrix}$$

So,

$$d = |\overrightarrow{AB}| = \sqrt{(-3+5t)^2 + (3-5t)^2 + (1-t)^2}$$

From GDC,

You could expand the expression to get the quadratic $d^2 = 51t^2 - 62t + 11$ and then complete the square or differentiate to find the minimum, but it is easier just to use your GDC

Min $d = 0.396$ km

and this occurs when $t = 0.608$ hours.

Exercise 2C

In questions 1 and 2, use the method demonstrated in Worked Example 2.15 to find the bearing on which the particle with given velocity vector is moving.

1 **a** $\mathbf{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

b $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

2 **a** $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

b $\mathbf{v} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$

In questions 3 and 4, use the method demonstrated in Worked Example 2.15 to find the speed of the particle moving with given velocity vector (all in ms^{-1}).

3 **a** $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$

b $\mathbf{v} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

4 **a** $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$

b $\mathbf{v} = \mathbf{i} + \mathbf{j} - 9\mathbf{k}$

In questions 5 to 8, use the method demonstrated in Worked Example 2.16 to find the velocity vector of an object moving in the given direction with the given speed.

5 **a** Direction $3\mathbf{i} - 4\mathbf{j}$ with speed 12 $\mathrm{m\,s}^{-1}$

b Direction $1.2\mathbf{i} + 0.5\mathbf{j}$ with speed 6.5 $\mathrm{m\,s}^{-1}$

6 **a** Direction $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ with speed 8 $\mathrm{m\,s}^{-1}$

b Direction $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ with speed 5 $\mathrm{m\,s}^{-1}$

7 **a** Direction $\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ with speed 10 $\mathrm{m\,s}^{-1}$

b Direction $2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ with speed 15 $\mathrm{m\,s}^{-1}$

8 **a** Direction $\begin{pmatrix} 0.4 \\ -0.3 \\ 1.2 \end{pmatrix}$ with speed 13 $\mathrm{m\,s}^{-1}$

b Direction $\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ with speed 2.1 $\mathrm{m\,s}^{-1}$

In questions 9 to 12, use the method demonstrated in Worked Example 2.17 to find the position vector of an object moving from initial position $\mathbf{r}_0$ m with velocity $\mathbf{v}$ m s^{-1} for time t s.

9 a $\mathbf{r}_0 = 2\mathbf{i} - 7\mathbf{j},\ \mathbf{v} = 0.6\mathbf{i} + 1.4\mathbf{j},\ t = 6$

b $\mathbf{r}_0 = \mathbf{i},\ \mathbf{v} = 3\mathbf{i} + 2.4\mathbf{j},\ t = 3$

10 a $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 3.2 \\ -5.5 \end{pmatrix},\ t = 10$

b $\mathbf{r}_0 = \begin{pmatrix} -3 \\ 4 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} -2 \\ 7 \end{pmatrix},\ t = 2.4$

11 a $\mathbf{r}_0 = -3\mathbf{i} + 2\mathbf{k},\ \mathbf{v} = 8\mathbf{i} - 3\mathbf{j} + 4\mathbf{k},\ t = 7.5$

b $\mathbf{r}_0 = 4\mathbf{i} + 0.5\mathbf{j} - 3.2\mathbf{k},\ \mathbf{v} = 8\mathbf{i} - 1.5\mathbf{j} + \mathbf{k},\ t = 5$

12 a $\mathbf{r}_0 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix},\ t = 2$

b $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 2.6 \\ -1.5 \\ 4.1 \end{pmatrix},\ t = 8$

13 A particle moves with constant velocity $\mathbf{v} = (0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$ m s^{-1}. At $t =$ seconds the particle is at the point with the position vector $(12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k})$ m.

a Find the speed of the particle.

b Write down an equation for the position vector of the particle at time t seconds.

c Does the particle pass through the point (16, 8, 14)?

14 Two particles move so that their position vectors at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 10 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0.5 \\ 2 \\ -0.5 \end{pmatrix}$$

The distance is measured in metres.

Find the distance between the particles when $t = 3$ seconds.

15 An object moves with a constant velocity. Its position vector at time t seconds is given by $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ where distance is measured in metres.

a Find the initial position of the object.

b Find the speed of the object.

c Find the distance of the object from the origin after 3 seconds.

16 Ship A is initially at the point with position vector $(9\mathbf{i} + \mathbf{j})$ km and moves with velocity $(4\mathbf{i} + 5\mathbf{j})$ km h^{-1}. Ship B is initially at the point with position vector $(-3\mathbf{i} + 2\mathbf{j})$ km and moves with velocity $(6\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors directed East and North respectively.

a Find expressions for the position vectors of A and B after t hours.

b Calculate the distance of A from B after 2 hours.

c Find the time at which A is due north of B.

17 In this question, the distance is measured in km and the time in hours.

An aeroplane, initially at the point (2, 0, 0), moves with constant speed 894 km h^{-1} in the direction of the vector $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

Find an equation for the position vector of the aeroplane at time t hours.

18 Two toy helicopters are flown, each in a straight line. The position vectors of the two helicopters at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Distance is measured in metres.

a Show that the paths of the helicopters cross.

b Determine whether the helicopters collide.

19 Two particles move so that their position vectors at time t are given by

$\mathbf{r}_1 = (1+2t)\mathbf{i} + (t-3)\mathbf{j} + (3+7t)\mathbf{k}$

and

$\mathbf{r}_2 = (9-2t)\mathbf{i} + (t-2)\mathbf{j} + (22+2t)\mathbf{k}$

a Find the speed of each particle.

b Determine whether the particles meet.

20 Particles P and Q are initially at the points $(-1, -1, 2)$ and $(7, -2, 5)$ respectively.

P moves with velocity $\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and Q with velocity $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$.

a Write down the position vector of each particle at time t.

b Show that their paths cross but that they do not collide.

21 In this question **i** and **j** are unit vectors directed East and North respectively. Time is measures in hours and distance in miles.

Two particles move with constant velocity. At $t = 0$, Particle 1 starts from the point $(3, 0)$ and moves with velocity $(-2\mathbf{i} + 5\mathbf{j})$; Particle 2 starts form the point $(0, 5)$ and moves with velocity $(4\mathbf{i} + \mathbf{j})$.

a Write down the position vector of each particle at time t.

b Find and simplify an expression for the distance between the two particles at time t.

c Hence find the minimum distance between the particles.

d Find the time at which particle 2 is due east of particle 1.

22 In this question time is measures in metres and distance in seconds.

Two model aeroplanes move with constant velocity. Plane A starts from the point $(12, -10, 1)$ and moves with velocity $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$ m s^{-1}; plane B starts form the point $(4, 1, 5)$ and moves with velocity $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ m s^{-1}.

a Write down the position vector of each particle at time t.

b Find an expression for the distance between the two particles at time t.

c Hence find the minimum distance between the particles.

23 The position vectors of two drones at time t hours are given by $\mathbf{r}_1 = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix}$

a **i** Show that the drones will collide.

ii Determine the position vector of the point of collision.

In fact, after half an hour, drone 2 changes its velocity vector to $\begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}$ to avoid the collision.

b Find the distance between the two drones at the time at which they would have collided.

24 Two flies move so that their position vectors at time t seconds are given by

$\mathbf{r}_1 = (0.56\mathbf{j} + 3\mathbf{k}) + t(1.2\mathbf{i} + 0.8\mathbf{j} - 0.1\mathbf{k})$

and $\mathbf{r}_2 = (3.96\mathbf{i} + \mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + 0.3\mathbf{k})$

where distance is measured in metres. The base vectors **i** and **j** are in the horizontal plane and vector **k** points upwards.

a Show that there is a time when one fly is vertically above the other.

b Find the distance between the flies at that time.

25 In this question, distance is measure in kilometres and time in hours.

A boat is moving with the constant velocity $(64\mathbf{i})$ km h^{-1}. At time $t = 0$ it is located at the origin. A small submarine is located at the point $(0, 0.5, -0.02)$. At time $t = 0$, it starts moving with a constant velocity in the direction of the vector $(40\mathbf{i} - 25\mathbf{i} + c\mathbf{k})$. Given that the submarine reaches the boat:

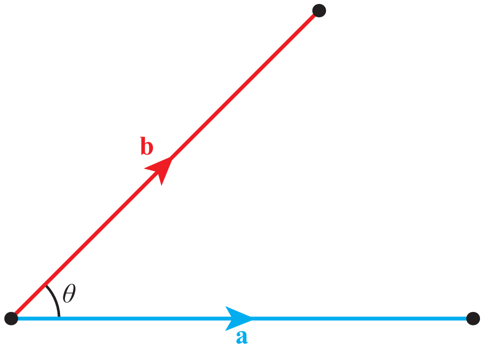
a Find the value of c.

b Find the speed of the submarine.

2D Scalar and vector product

Definition and calculation of the scalar product of two vectors

The diagram shows two lines with angle θ between them. **a** and **b** are vectors in the directions of the two lines. Notice that both arrows are pointing away from the intersection point.



It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors.

Links to: Physics

The formula for the scalar product can be considered as the projection of one vector onto the other and it has many applications in physics. For example, if a force **F** acts on an object that moves from the origin to a point with position x, then the work done is $\mathbf{F} \bullet \boldsymbol{x}$.

KEY POINT 2.12

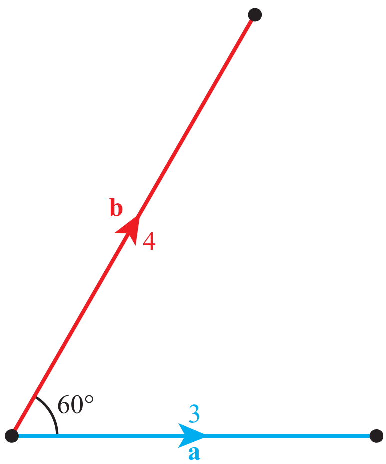
The **scalar product** (or **dot product**) of two vectors is defined by

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**.

WORKED EXAMPLE 2.19

The diagram shows vectors **a** and **b**. The numbers represent their lengths. Find the value of $\mathbf{a} \bullet \mathbf{b}$.



Use the formula for $\mathbf{a} \bullet \mathbf{b}$ ······ $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| \bullet |\mathbf{b}|\cos\theta$

The magnitudes are the lengths of the lines ······ $= 3 \times 4 \times \cos 60°$

$= 6$

> Later in this section you will meet the cross product, for which the result is a vector.

Notice that the scalar product is a number (scalar).

Vectors are often given in terms of components, rather than by magnitude and direction. You can use the cosine rule to express the scalar product in terms of the components of the two vectors.

KEY POINT 2.13

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Tip

The value of the scalar product can be negative.

WORKED EXAMPLE 2.20

Given that $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$, calculate $\mathbf{a} \cdot \mathbf{b}$.

Use $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\mathbf{a} \cdot \mathbf{b} = (3)(-3) + (-2)(1) + (1)(4)$$
$$= -9 - 2 + 4$$
$$= -7$$

The angle between two vectors

Combining the results from Key Points 8.10 and 8.11 gives a formula for calculating the angle between two vectors given in component form.

Tip

The same formula can be used to find the angle between vectors in two dimensions – just set $a_3 = b_3 = 0$.

KEY POINT 2.14

If θ is the angle between vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

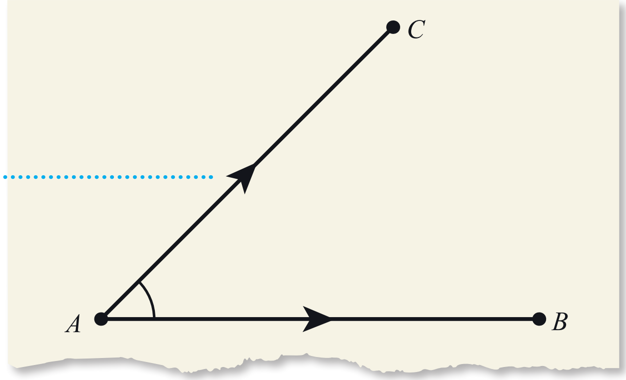
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

WORKED EXAMPLE 2.21

Given points $A(3, -5, 2)$, $B(4, 1, 1)$ and $C(-1, 1, 2)$, find the size of angle $B\hat{A}C$ in degrees.

It is always a good idea to draw a diagram to see which vectors you need to use



You can see that the required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$.

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

You need to find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{1\times(-4) + 6\times 6 + (-1)\times 0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\theta = \cos^{-1}(0.7199) = 44.0°$$

Be the Examiner 2.1

Given points $A(-1, 4, 2)$, $B(3, 3, 1)$ and $C(2, -5, 3)$, find the size of angle $A\hat{B}C$ in degrees.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$
$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$
$\cos\theta = \frac{-4+8-2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$
$= 0.142$	$= -0.142$	$= -0.142$
$\theta = \cos^{-1}(0.142) = 81.8°$	$\theta = \cos^{-1}(-0.142) = 98.2°$	$\theta = \cos^{-1}(-0.142) = 98.2°$
	angle $= 180 - 98.2 = 81.8°$	

TOOLKIT: Problem Solving

There is more than one solution to $\cos x = 0.7199$ in the worked example above, but we have only given one answer. What do the other solutions represent?

Perpendicular and parallel vectors

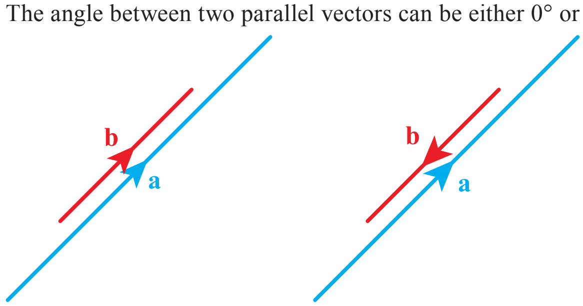
Two further important properties of the scalar product concern perpendicular and parallel vectors. They are derived using the facts that $\cos 90° = 0$, $\cos 0° = 1$ and $\cos 180° = -1$.

KEY POINT 2.15

- If **a** and **b** are perpendicular vectors, then $\mathbf{a} \cdot \mathbf{b} = 0$.
- If **a** and **b** are parallel vectors, then $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Tip

The angle between two parallel vectors can be either 0° or 180°.



WORKED EXAMPLE 2.22

Find the value of t such that the vector $\mathbf{a} = \begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix}$ is perpendicular to the vector $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

For perpendicular vectors, $\mathbf{a} \cdot \mathbf{b} = 0$

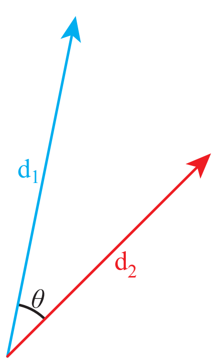
$$\begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 0$$

Express the scalar product in terms of the components

$$(3t)(1) + (1+t)(-2) + (2-5t)(2) = 0$$

$$-9t + 2 = 0$$

$$t = \frac{2}{9}$$



Angle between two lines

You can find the angle between two lines by using their direction vectors.

KEY POINT 2.16

The angle between two lines is equal to the angle between their direction vectors.

WORKED EXAMPLE 2.23

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Direction vectors:

Identify the direction vectors

$$\mathbf{d}_1 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix},\ \mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Use scalar product to find the angle between the direction vectors

$$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}$$

$$= \frac{(-4+0-6)}{\sqrt{4^2+0^2+3^2}\sqrt{1^2+1^2+2^2}}$$

$$= -0.816$$

$$\theta = \cos^{-1}(-0.816) = 144.7°$$

The question asks for the acute angle

$$180 - 144.7 = 35.3°$$

Be the Examiner 2.2

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{d}_1 \bullet \mathbf{d}_2 = 20+0+6 = 26$	$\mathbf{d}_1 \bullet \mathbf{d}_2 = -4-10+3 = -11$	$\mathbf{d}_1 \bullet \mathbf{d}_2 = -4-10+3 = -11$
$\|\mathbf{d}_1\| = \sqrt{25+0+4} = \sqrt{29}$	$\|\mathbf{d}_1\| = \sqrt{1+4+9} = \sqrt{14}$	$\|\mathbf{d}_1\| = \sqrt{1+4+9} = \sqrt{14}$
$\|\mathbf{d}_2\| = \sqrt{16+1+9} = \sqrt{26}$	$\|\mathbf{d}_2\| = \sqrt{16+25+1} = \sqrt{42}$	$\|\mathbf{d}_2\| = \sqrt{16+25+1} = \sqrt{42}$
$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\|\mathbf{d}_1\|\|\mathbf{d}_2\|}$	$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\|\mathbf{d}_1\|\|\mathbf{d}_2\|}$	$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\|\mathbf{d}_1\|\|\mathbf{d}_2\|}$
$= \frac{26}{\sqrt{29\times 26}} = 0.947$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$
$\theta = 18.8°$	$\theta = 117°$	$\theta = 117°$
	So, acute angle $= 180 - 117 = 63.0°$	So, acute angle $= 117 - 90 = 27.0°$

Sample pages not final

Tip

Remember that you can use the scalar product to identify perpendicular vectors. This is particularly useful for finding perpendicular distances.

WORKED EXAMPLE 2.24

Line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, –2).

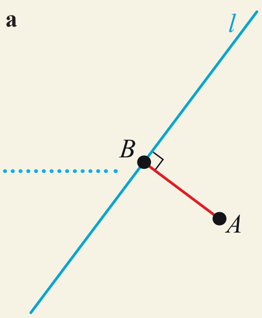
Point B lies on the line l and AB is perpendicular to l.

a Find the coordinates of B.

b Hence, find the shortest distance from A to l.

a

Draw a diagram. The line AB should be perpendicular to the direction vector of l



$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

We know that B lies on l, so its position vector is given by the equation for $\mathbf{r}$

$$\mathbf{b} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ t \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$= \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

We can now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$(\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\lambda = -4$$

Using this value of λ in the equation of the line gives the position vector of B

$$\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

B has coordinates (–1, 3, –4)

b

The shortest distance from a point to a line is the perpendicular distance, in other words, the distance AB

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{16+36+4} = 2\sqrt{14}$$

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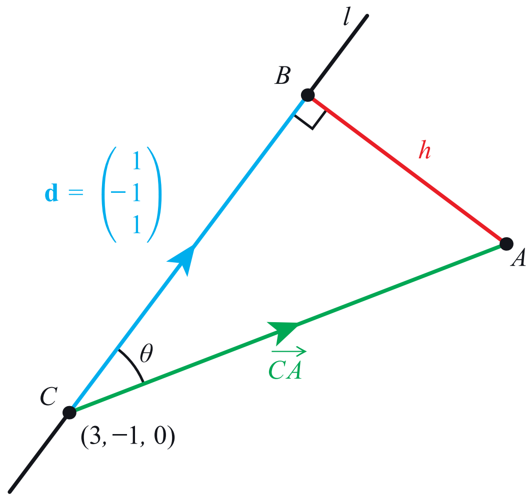
TOOLKIT: Problem Solving

Here are two alternative methods to solve the problem in Worked Example 8.27.

1 You found that $\overrightarrow{AB} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$. Show that $AB^2 = 3\lambda^2 + 24\lambda + 104$.

Hence, find the smallest possible value of AB.

2 In the diagram below, h is the required shortest distance from A to the line. You know, from the equation of the line, that the point $C(3, -1, 0)$ lies on the line. θ is the angle between $\overrightarrow{CA}$ and the direction vector of the line.



a Find the length AC.

b Find the exact value of $\cos\theta$. Hence, show that $\sin\theta = \frac{6}{\sqrt{78}}$.

c Use the right-angled triangle ABC to find the length h.

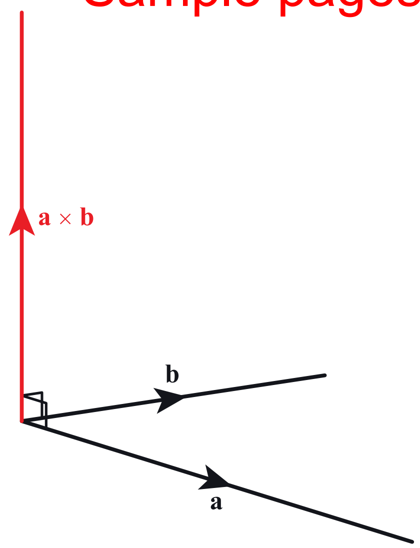
The definition and calculation of the vector product of two vectors

One way to define the vector product is to give its magnitude and direction.

KEY POINT 2.17

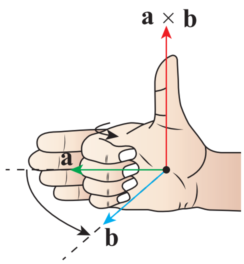
The **vector product** (or **cross product**) of vectors **a** and **b** is a vector denoted by $\mathbf{a} \times \mathbf{b}$.

- The magnitude is equal to $|\mathbf{a}||\mathbf{b}|\sin\theta$ (where θ is the angle between **a** and **b**).
- The direction is perpendicular to both **a** and **b** (as shown in the diagram).



Tip

The direction of the vector $\mathbf{a} \times \mathbf{b}$ is given by the right-hand screw rule.

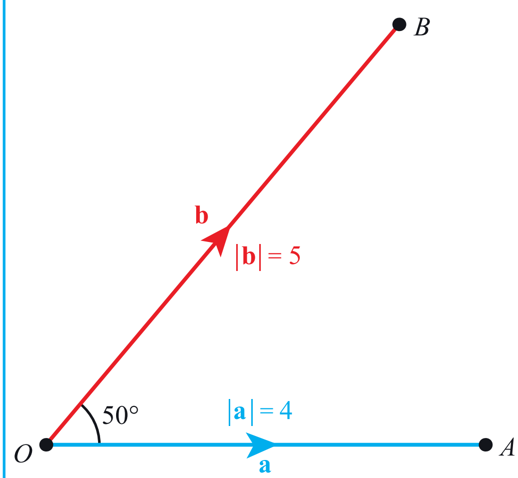


Links to: Physics

Vectors are used to model quantities in both mathematics and physics. In geometry, the vector product is used to find the direction which is perpendicular to two given vectors. In physics, it is used in many equations involving quantities which are modelled as vectors. For example, the angular momentum of a particle moving in a circle is given by $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ is the position vector and $\mathbf{v}$ the velocity of the particle. In electrodynamics, the Lorentz force acting on a charge q moving with velocity $\boldsymbol{v}$ in magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

WORKED EXAMPLE 2.25

For the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the diagram, find the magnitude of $\mathbf{a} \times \mathbf{b}$.



Use $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$|\mathbf{a} \times \mathbf{b}| = (4)(5)\sin 50°$$
$$= 15.3$$

The vector product can also be expressed in component form.

KEY POINT 2.18

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$.

Tip

This formula is given in the Mathematics: applications and interpretation formula booklet.

WORKED EXAMPLE 2.26

Given that $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, find the vector $\mathbf{a} \times \mathbf{b}$.

Use the formula for the component form of $\mathbf{a} \times \mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (-3)(5) - (2)(1) \\ (2)(2) - (1)(5) \\ (1)(1) - (-3)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$$

You can check that $\begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

Be the Examiner 2.3

Find $\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\begin{pmatrix} 6-5 \\ 20+6 \\ 8+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 26 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 20-6 \\ 8-2 \\ 6-5 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 20+6 \\ -(8+2) \\ 6-5 \end{pmatrix} = \begin{pmatrix} 26 \\ -10 \\ 1 \end{pmatrix}$

It is worth remembering the special result for parallel and perpendicular vectors, which follows from the fact that $\sin 0° = \sin 180° = 0$ and $\sin 90° = 1$.

Tip

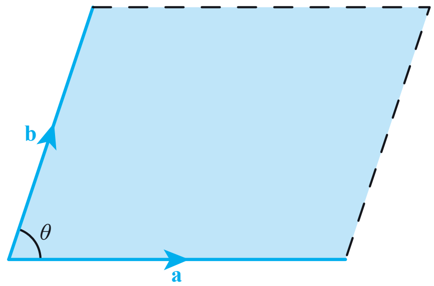
Notice that, since the vector product produces a vector, each zero in Key Point 2.19 is the zero *vector*, not a scalar value.

KEY POINT 2.19

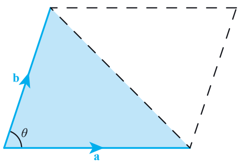
- If **a** and **b** are parallel vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- If **a** and **b** are perpendicular vectors, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Geometric interpretation

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}| \sin\theta$. But this is also the area of the parallelogram determined by the vectors **a** and **b**.



A parallelogram can be divided into two triangles, so you can also use the vector product to find the area of a triangle.



KEY POINT 2.20

The area of the triangle with two sides defined by vectors **a** and **b** is equal to $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

When you are given the coordinates of the vertices of a triangle, sketch a diagram to identify which two vectors to use.

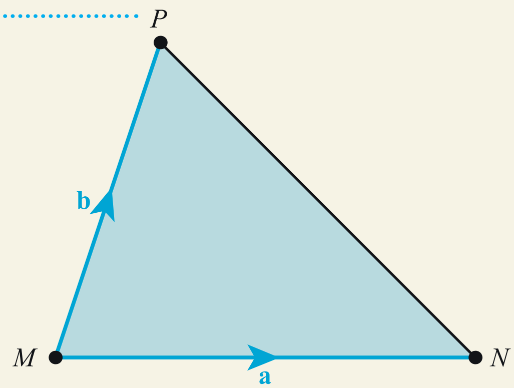
Tip

It does not matter which two sides of the triangle you use.

WORKED EXAMPLE 2.27

Find the area of the triangle with vertices $M(1, 4, 2)$, $N(3, -3, 0)$ and $P(-1, 8, 9)$.

Sketch a diagram to see which vectors to use



Two of the sides of the triangle are vectors $\overrightarrow{MN}$ and $\overrightarrow{MP}$

$$\mathbf{a} = \overrightarrow{MN} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{MP} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

The area of the triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. Find the vector $\mathbf{a}\times\mathbf{b}$ first…

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -49+8 \\ 4-14 \\ 8-14 \end{pmatrix} = \begin{pmatrix} -41 \\ -10 \\ -6 \end{pmatrix}$$

… then find half of its magnitude

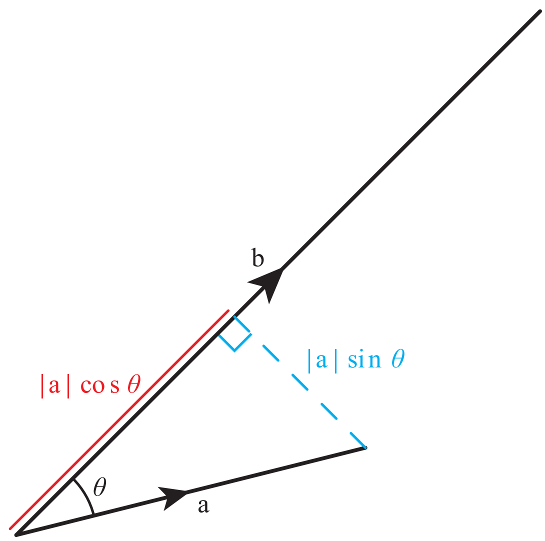
$$|\mathbf{a}\times\mathbf{b}| = \sqrt{41^2+10^2+6^2} = 42.6$$

$$\text{Area} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = 21.3$$

Components of vectors

A vector is given in terms of components in the direction of the base vectors. For example, given the force $\mathbf{F} = (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$N, we know that a 2N force acts in the direction of the vector $\mathbf{i}$, a 3N force acts in the direction of the vector $-\mathbf{j}$ and a 4N force acts in the direction of the vector $\mathbf{k}$.

However, often we want to know what the component of a vector is in a direction of a vector that is not one of the base vectors. We can do this using trigonometry:



So, the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $|\mathbf{a}|\cos\theta$.

We can also see that the component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ is $|\mathbf{a}|\sin\theta$.

Since $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ we can express the results shown above in terms of the dot and cross product respectively.

KEY POINT 2.21

The component of $\mathbf{a}$ acting

- in the direction of $\mathbf{b}$ is $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$
- perpendicular to $\mathbf{b}$ is $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

WORKED EXAMPLE 2.28

A tugboat pulls a barge into port. The tugboat starts pulling with a force of $\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix}$ kN and the direction back to port initially is $\begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}$.

Find the component of the force in the direction back to port.

The component of the force, F, acting in the direction of the port d is $\frac{\mathbf{F} \bullet \mathbf{d}}{|\mathbf{d}|}$

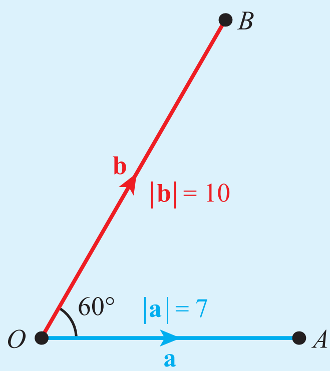
The component of $\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix}$ in the direction of $\begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}}{\left|\begin{matrix} 11 \\ 4 \\ 0 \end{matrix}\right|} = \frac{108}{\sqrt{11^2 + 4^2 + 0^2}} = 9.23\text{kN}$$

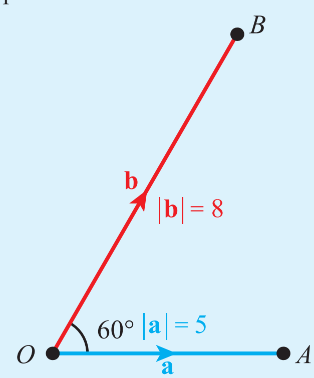
Exercise 2D

For questions 1 to 3, use the method demonstrated in Worked Example 2.19 to find **a** • **b** for the vectors in each diagram.

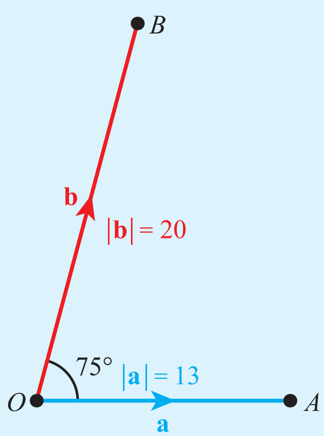
1 **a**



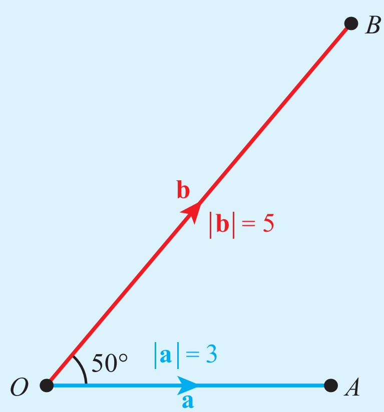
b



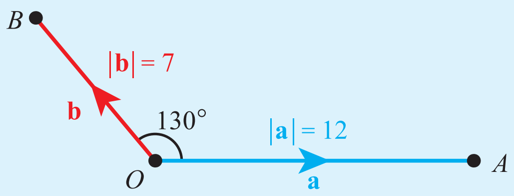
2 **a**



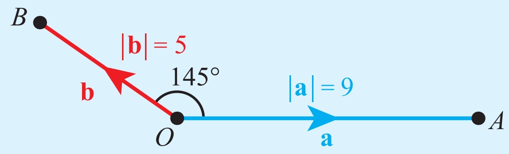
b



3 **a**



b



For questions 4 to 6, use the method demonstrated in Worked Example 2.20 to find $\mathbf{a} \cdot \mathbf{b}$ for the two given vectors.

4 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -12 \\ 4 \\ -8 \end{pmatrix}$

5 **a** $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$

6 **a** $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ **b** $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$

For questions 7 to 10, use the method demonstrated in Worked Example 2.21 to find the required angle, giving your answer to the nearest degree.

7 **a** Angle between vectors $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ **b** Angle between vectors $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

8 **a** Angle between vectors $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ **b** Angle between vectors $\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$

9 **a** Angle $B\hat{A}C$ where $A(2, 1, 0)$, $B(3, 1, 2)$, $C(4, 4, 1)$ **b** Angle $B\hat{A}C$ where $A(2, 1, 0)$, $B(3, 0, 0)$, $C(2, -2, 4)$

10 **a** Angle $A\hat{B}C$ where $A(3, 6, 5)$, $B(2, 3, 6)$, $C(4, 0, 1)$ **b** Angle $A\hat{B}C$ where $A(8, -1, 2)$, $B(3, 1, 2)$, $C(0, -2, 0)$

For questions 11 to 14, use the method demonstrated in Worked Example 2.22 to find the value t such that the two given vectors are perpendicular.

11 **a** $\begin{pmatrix} t+1 \\ 2t-1 \\ 2t \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

12 **a** $\begin{pmatrix} t+1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ t \\ 2 \end{pmatrix}$

b $\begin{pmatrix} 1 \\ t+1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \\ 3-7t \end{pmatrix}$

13 **a** $(5-t)\mathbf{i} + 3\mathbf{j} - (10-t)\mathbf{k}$ and $-3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

b $(2t)\mathbf{i} + (t+1)\mathbf{j} - 5\mathbf{k}$ and $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

14 **a** $5t\mathbf{i} - (2+t)\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - t\mathbf{k}$

b $t\mathbf{i} - 3\mathbf{k}$ and $2\mathbf{i} + (t+4)\mathbf{j}$

15 **a** $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

16 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}$

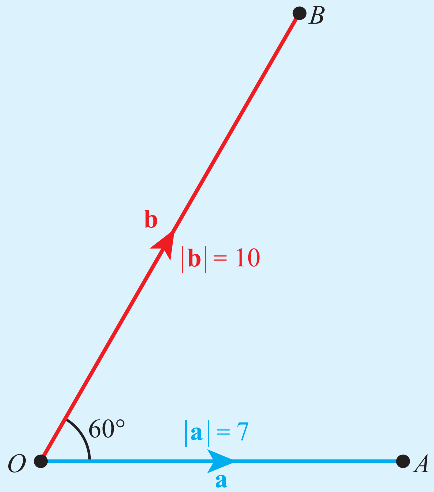
b $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

17 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

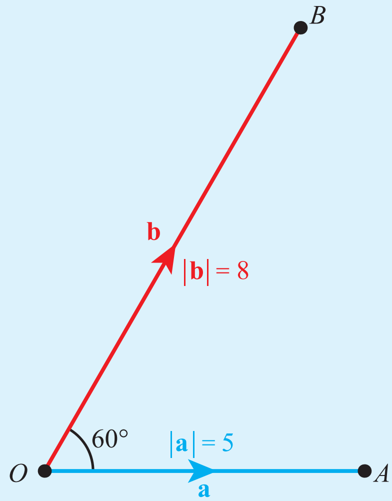
b $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For questions 18 to 20, use the method demonstrated in Worked Example 8.32 to find the magnitude of $\mathbf{a} \times \mathbf{b}$ for the vectors in each diagram.

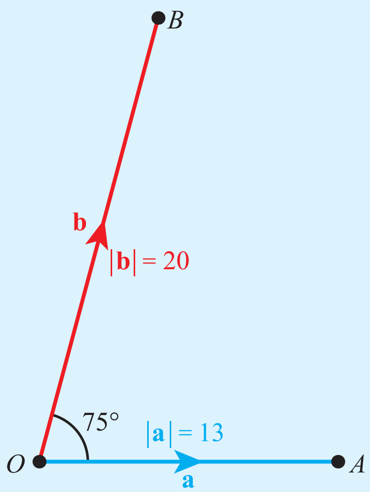
18 a



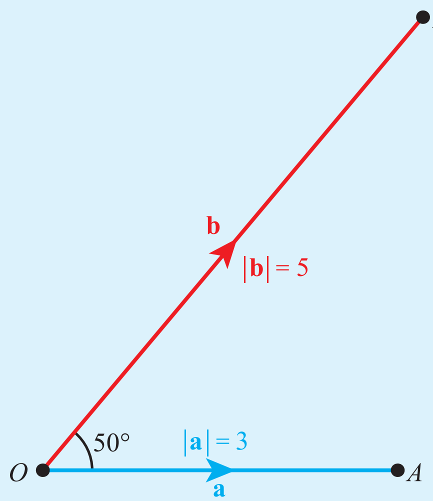
b



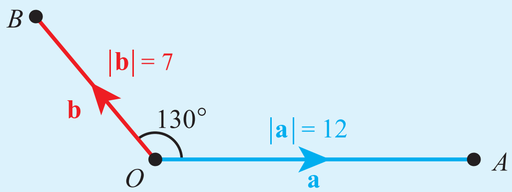
19 a



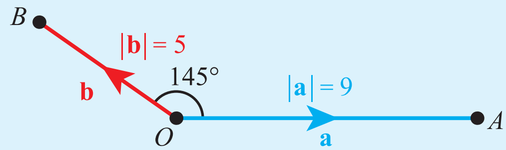
b



20 a



b



For questions 21 to 23, use the method demonstrated in Worked Example 2.26 to find $\mathbf{a} \times \mathbf{b}$ for the two given vectors.

21 a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

22 a $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

23 a $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ b $\mathbf{a} = -3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$

For questions 24 to 26, use the method demonstrated in Worked Example 2.27 to find the area of the triangle with given vertices.

24 a $(1, 3, 3)$, $(-1, 1, 2)$ and $(1, -2, 4)$

b $(3, -5, 1)$, $(-1, 1, 3)$ and $(-1, -5, 2)$

25 a $(-3, -5, 1)$, $(4, 7, 2)$ and $(-1, 2, 2)$

b $(4, 0, 2)$, $(4, 1, 5)$ and $(4, -3, 2)$

26 a $(1, 5, 2)$, $(8, 4, 6)$ and $(0, 6, 7)$

b $(2, 1, 2)$, $(3, 8, 4)$ and $(1, 3, -1)$

In questions 27 to 32, use the method demonstrated in Worked Example 2.28 to find the component of the vector v in the given direction:

27 a $v = 3\mathbf{i} - 2\mathbf{j}$ in the direction of $4\mathbf{i} + 5\mathbf{j}$ b $v = 5\mathbf{i} + 2\mathbf{j}$ in the direction of $3\mathbf{i} - \mathbf{j}$

28 a $v = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ in the direction of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ b $v = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$ in the direction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

29 a $v = 7\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ in the direction of $3\mathbf{j} - 4\mathbf{k}$ b $v = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$

30 a $v = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ in the direction of $\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ b $v = \begin{pmatrix} -2 \\ 7 \\ 1 \end{pmatrix}$ in the direction of $\begin{pmatrix} 3 \\ 0 \\ 8 \end{pmatrix}$

31 a $v = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ b $v = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$ perpendicular to $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

32 a $v = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ perpendicular to $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ b $v = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$ perpendicular to $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$

33 Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$.

34 Four points are given with coordinates $A(2, -1, 3)$, $B(1, 1, 2)$, $C(6, -1, 2)$ and $D(7, -3, 3)$. Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

35 Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \bullet \mathbf{b} = 10$ find, in degrees, the angle between $\mathbf{a}$ and $\mathbf{b}$.

36 Given that $|\mathbf{c}| = 9$, $|\mathbf{d}| = 12$ and $\mathbf{c} \bullet \mathbf{d} = -15$ find, in degrees, the angle between $\mathbf{c}$ and $\mathbf{d}$.

37 Given that $|\mathbf{a}| = 8$, $\mathbf{a} \bullet \mathbf{b} = 12$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$ find the value of $|\mathbf{b}|$.

38 Find the acute angle between lines with equations $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

39 Show that the lines with equations $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ are perpendicular.

40 Given that $|\mathbf{a}| = 5$, $|\mathbf{b}| = 7$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $30°$ find the exact value of $|\mathbf{a} \times \mathbf{b}|$.

41 Given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $|\mathbf{a} \times \mathbf{b}| = 7$ find, in radians, the acute angle between the directions of vectors $\mathbf{a}$ and $\mathbf{b}$.

42 Given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 1$ and $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ find, in radians, the acute angle between the directions of vectors $\mathbf{a}$ and $\mathbf{b}$.

43 Find a vector perpendicular to both $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

44 A rowing boat is travelling at a speed of $7\,\text{km h}^{-1}$ in the direction $\begin{pmatrix}4\\1\end{pmatrix}$.

The river is flowing at $10\,\text{km h}^{-1}$ in the direction $\begin{pmatrix}2\\3\end{pmatrix}$.

a Write the velocity of the river as a column vector.

b Find the component of the water's velocity acting in the direction the boat is travelling.

45 Find, in degrees, the angles of the triangle with vertices $(1, 1, 3)$, $(2, -1, 1)$ and $(5, 1, 2)$.

46 The vertices of a triangle have position vectors $\mathbf{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}4\\-1\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}7\\1\\-2\end{pmatrix}$.
Find, in degrees, the angles of the triangle.

47 Vertices of a triangle have position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i}$.

a Show that the triangle is right angled.

b Calculate the other two angles of the triangle.

c Find the area of the triangle.

48 Given that $\mathbf{p}$ is a unit vector making a 45° angle with vector $\mathbf{q}$, and that $\mathbf{p} \bullet \mathbf{q} = 3\sqrt{2}$, find $|\mathbf{q}|$.

49 Given that $\mathbf{p} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}2\\1\\1\end{pmatrix}$ find the value of the scalar t such that $\mathbf{p} + t\mathbf{q}$ is perpendicular to $\begin{pmatrix}3\\5\\1\end{pmatrix}$.

50 Given that the vectors $t\mathbf{i} - 3\mathbf{k}$ and $2t\mathbf{i} + \mathbf{j} + t\mathbf{k}$ are perpendicular, find the possible values of t.

51 A line has parametric equation

$$x = \frac{1}{2} + \frac{3}{2}\lambda,\ y = 7,\ z = 2 - 4\lambda$$

a Find a vector equation of the line.

b Find the angle that the line makes with the x-axis

52 Find, in degrees, the acute angle between the lines

$$\begin{cases} x = 3 + 5\lambda \\ y = 2 + \lambda \\ z = 3 - 2\lambda \end{cases} \quad \text{and} \quad \begin{cases} x = 3\mu - 1 \\ y = 1 \\ z = 3 - \mu \end{cases}$$

53 Find a unit vector perpendicular to both $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

54 Find a unit vector perpendicular to both $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\5\end{pmatrix}$.

55 a Explain why $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a} = 0$.

b Evaluate $(\mathbf{a} \times \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$.

56 Given points A, B and C with coordinates $(3, -5, 1)$, $(7, 7, 2)$ and $(-1, 1, 3)$

a calculate $\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\mathbf{p} = \overrightarrow{BA} \times \overrightarrow{BC}$.

b What can you say about vectors $\mathbf{p}$ and $\mathbf{q}$?

57 A quadrilateral has vertices $A(1, 4, 2)$, $B(3, -3, 0)$, $C(1, 1, 7)$ and $D(-1, 8, 9)$.

a Show that the quadrilateral is a parallelogram.

b Find the area of the quadrilateral.

58 Find the area of the triangle with vertices $(2, 1, 2)$, $(5, 0, 1)$ and $(-1, 3, 5)$.

59 The points $A(3, 1, 2)$, $B(-1, 1, 5)$ and $C(7, -2, 3)$ are vertices of a parallelogram $ABCD$.

a Find the coordinates of D.

b Calculate the area of the parallelogram.

60 Prove that for any two vectors **a** and **b**, $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \bullet \mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$.

61 Times run by athletes in a 100 m sprint race can only be counted as official records if the wind speed in the direction of the track and assisting the runners is less than $2\,\mathrm{m\,s^{-1}}$.

A track is in the direction $\begin{pmatrix}1\\2\end{pmatrix}$ and a wind of $3\,\mathrm{m\,s^{-1}}$ is blowing in the direction $\begin{pmatrix}3\\4\end{pmatrix}$.

Determine whether a record time run in these conditions is valid or not.

62 $ABCD$ is a parallelogram with $AB \parallel DC$. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

a Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of **a** and **b**.

b Simplify $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{b} - \mathbf{a})$.

c Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular.

63 Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

a Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

b Find the value of λ for which CBA is a right angle.

c For the value of λ found above, calculate the exact distance from C to the line OA.

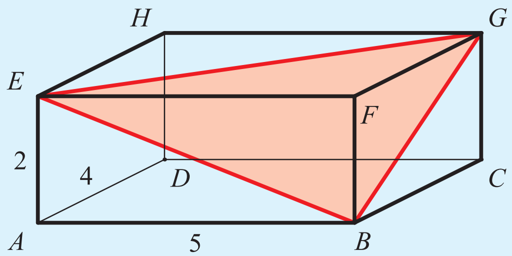
64 Given line $l : \mathbf{r} = \begin{pmatrix}5\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\3\end{pmatrix}$ and point $P(21, 5, 10)$,

a find the coordinates of point M on l such that PM is perpendicular to l

b show that the point $Q(15, -14, 17)$ lies on l

c find the coordinates of point R on l such that $|PR| = |PQ|$.

65 A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are $A(0, 0, 0)$, $B(5, 0, 0)$, $D(0,4, 0)$ and $E(0, 0, 2)$.



a Find the coordinates of the reMAAning four vertices.

Face diagonals BE, BG and EG are drawn as shown.

b Find the area of the triangle BEG.

Checklist

- You should understand the concept of a vector and of a scalar.
  - A vector is a quantity that has both magnitude and direction.
  - A scalar is a quantity that has only magnitude but no direction.
- You should know about different ways of representing vectors and how to add, subtract and multiply vectors by a scalar.
  - Vectors can be represented either as directed line segments or by their components (as column vectors or using **i**, **j**, **k** base vectors).
- You should be able to find the resultant of two or more vectors
- You should be able to identify when vectors are parallel.
  - If vectors **a** and **b** are parallel then $\mathbf{b} = t\mathbf{a}$ for some scalar t.
- You should be able to find the magnitude of a vector.
  - The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- You should be able to find a unit vector in a given direction.
  - The unit vector in the same direction as vector **a** is $\frac{1}{|\mathbf{a}|}\mathbf{a}$.
- You should know about position vectors and displacement vectors.
  - The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, where O is the origin. The components of **a** are the coordinates of A.
  - If points A and B have position vectors **a** and **b** then:
    - — the displacement vector from A to B is $\overrightarrow{AB} = \overrightarrow{OB} = \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$
    - — the distance between the points A and B is $AB = |\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$.
- You should be able to find the vector equation of a line in two and three dimensions and convert to parametric form.
  - The vector equation of a line has the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where **d** is the direction vector and **a** is the position vector of one point on the line.
  - The components of the position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are the coordinates of a general point on the line.
  - The parametric form of the equation of a line is found by expressing x, y and **z** in terms of λ.
- You should be able to determine whether two lines intersect and find the point of intersection.
- You should be able to model linear motion with constant velocity in two and three dimensions.
  - An object with constant velocity **v** is:
    - — moving parallel to the vector **v** with speed $|\mathbf{v}|$
    - — has position vector after time t given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is its initial position vector.
- You should be able to use the scalar product to find the angle between two vectors.
  - If θ is the angle between vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
  - where $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- You should be able to identify when two vectors are perpendicular.
  - If **a** and **b** are perpendicular vectors then $\mathbf{a} \bullet \mathbf{b} = 0$
- You should be able to find the angle between two lines.
  - The angle between two lines is equal to the angle between their direction vectors.

- You should be able to use the vector product to find perpendicular directions and areas.
  - The vector product (cross product) is a vector perpendicular to both **a** and **b**.
  - The component form is $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.
  - The magnitude of the cross product is $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, which equals the area of the parallelogram formed by the vectors **a** and **b**.
  - If **a** and **b** are parallel vectors then $\mathbf{a}\times\mathbf{b} = 0$.
- You should be able to find components of vectors in given directions
  - The component of **a** acting
    - in the direction of **b** is $\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}|\cos\theta$
    - perpendicular to **b** is $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a}|\sin\theta$.

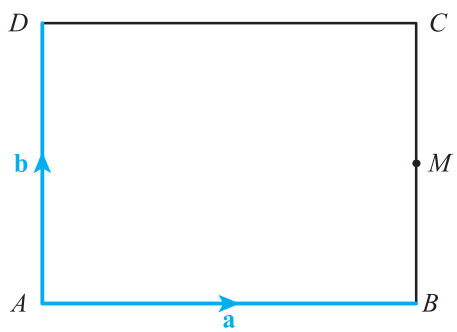
Mixed Practice

1 An aeroplane takes off from an airport and travels with displacement vector $(-22\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})$ km. It then changes direction and travels with displacement vector $(215\mathbf{i} - 73\mathbf{j} - 0.5\mathbf{k})$ km, and then changes direction again and travels with displacement vector $(83\mathbf{i} + 16\mathbf{j})$ km.
Find

a the displacement vector it now needs to travel along to return to the airport

b the distance it needs to travel to return to the airport.

2 The diagram shows a rectangle $ABCD$, with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of BC.



a Express $\overrightarrow{MD}$ in terms of **a** and **b**.

b N is the midpoint of DM. Express $\overrightarrow{AN}$ in terms of **a** and **b**.

c P is the point on the extension of the side BC such that $CP = CM$. Show that A, N and P lie on the same straight line.

3 Points A, B and D have coordinates (1, 1, 7), (−1, 6, 3) and (3, 1, k), respectively. AD is perpendicular to AB.

a Write down, in terms of k, the vector $\overrightarrow{AD}$.

b Show that $k = 6$.
Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.

c Find the coordinates of C.

d Find the exact value of $\cos(A\hat{D}C)$

4 Line l_1 has equation $\mathbf{r} = \begin{pmatrix}5\\1\\2\end{pmatrix} + t\begin{pmatrix}-1\\1\\3\end{pmatrix}$ and line l_2 has equation $\mathbf{r} = \begin{pmatrix}5\\4\\9\end{pmatrix} + s\begin{pmatrix}2\\1\\1\end{pmatrix}$.

Find the coordinates of the point of intersection of the two lines.

5 **a** Show that the lines

$$\mathbf{r} = \begin{pmatrix}2\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\2\end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix}-2\\0\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\-3\end{pmatrix} \text{ are perpendicular.}$$

b Determine whether the lines intersect.

6 Show that the lines

$$\mathbf{r} = \begin{pmatrix}4\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\1\end{pmatrix} \quad \text{and} \quad \begin{cases} x = 1 + 4\mu \\ y = -2 + 3\mu \\ z = 0.5 + 2\mu \end{cases}$$

do not intersect.

7 **a** Find a vector equation of the line through the points $A(1, -3, 2)$ and $B(2, 2, 1)$.

b Find the acute angle between this line and the line l_2 with equation $\mathbf{r} = \begin{pmatrix}4\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\7\end{pmatrix}$

c Find the value of k for which the point $C(7, 3, k)$ lies on l_2.

d Find the distance AC.

8 **a** Find the vector equation of the line with parametric equation $x = 3\lambda + 1$, $y = 4 - 2\lambda$, $z = 3\lambda - 1$.

b Find the unit vector in the direction of the line.

9 In this question, distance is measured in metres and time in seconds. The base vectors **i**, **j** and **k** point East, North and up, respectively.

An aeroplane takes off from the ground. It moves with constant velocity $\mathbf{v} = (116\mathbf{i} + 52\mathbf{j} + 12\mathbf{k})$.

a Find the speed of the aeroplane.

b How long does it take for the aeroplane to reach a height of 1 km?

10 In this question **i** and **j** are unit vectors directed East and North respectively.

Ship A is initially at the point with position vector $\begin{pmatrix}2\\-8\end{pmatrix}$ km and moves with velocity $\begin{pmatrix}-5\\3\end{pmatrix}$ km h^{-1}.

Ship B is initially at the point with position vector $\begin{pmatrix}3\\4\end{pmatrix}$ km and moves with velocity $\begin{pmatrix}2\\-6\end{pmatrix}$ km h^{-1}.

a Find expressions for the position vectors of A and B after t hours.

b Calculate the distance of A from B after 3 hours.

c Find the time at which A is due west of B.

11 In this question the base vectors **i**, **j** and **k** point East, North and up, respectively.

A drone takes off from a point with position vector $(2\mathbf{i} - 3\mathbf{j})$ m and moves with constant velocity $(10\mathbf{i} - 9\mathbf{j} + 7\mathbf{k})\,\mathrm{m\,s^{-1}}$

a Write down an expression for the position vector of the drone after time t seconds.

b Find the speed of the drone.

The drone's controller is located at the point with position vector $(67\mathbf{i} - 61.5\mathbf{j})$ m.

c Find the time at which it is directly overhead of the controller.

12 Find the value of x, with $0 < x < \frac{\pi}{2}$, such that the vectors $\begin{pmatrix} 3\sin x \\ 8 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4\cos x \\ 1 \\ -2 \end{pmatrix}$ are perpendicular.

13 Given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\mathbf{d} = 4\mathbf{i} - \mathbf{j} + p\mathbf{k}$,

a find $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

b find the value of p such that $\mathbf{d}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$.

14 Points A and C have position vectors $\mathbf{a} = 2\mathbf{j} - 5\mathbf{k}$ and $\mathrm{c} = \mathbf{i} + 3\mathbf{k}$

a Find $\overrightarrow{OA} \times \overrightarrow{OC}$.

b Find the coordinates of the point B such that $OABC$ is a parallelogram.

c Find the exact area of $OABC$.

15 A javelin thrower aims to throw the javelin in the direction $\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. He releases the javelin with velocity $\begin{pmatrix} -2 \\ 21 \\ 15 \end{pmatrix}\mathrm{m\,s^{-1}}$.

Find

a the speed of the javelin at the point of release

b the component of the velocity in the direction he was aiming

c the component of the velocity perpendicular to the direction he was aiming.

16 The position vectors of the points A, B and C are **a**, **b** and **c** respectively, relative to an origin O. The following diagram shows the triangle ABC and points M, R, S and T.

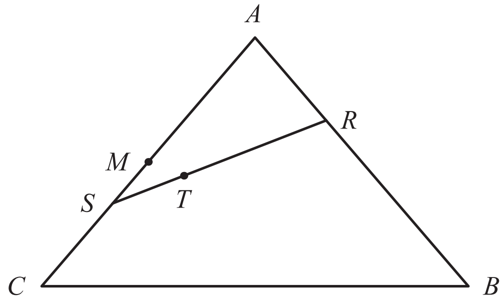


diagram not to scale

M is the mid-point of $[AC]$.

S is a point on $[AC]$ such that $AS = \frac{2}{3}AC$.

R is a point on $[AB]$ such that $AR = \frac{1}{3}AB$.

T is a point on $[RS]$ such that $RT = \frac{2}{3}RS$.

a **i** Express $\overrightarrow{AM}$ in terms of **a** and **c**.

ii Hence show that $\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}$.

b **ii** Express $\overrightarrow{RA}$ in terms of **a** and **b**.

ii Show that $\overrightarrow{RT} = \frac{2}{9}\mathbf{a} - \frac{2}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}$.

c Prove that T lies on $[BM]$.

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17 Three forces, $\mathbf{F}_1 = (5\mathbf{i} + a\mathbf{k})$ N, $\mathbf{F}_2 = (b\mathbf{i} - 7\mathbf{j} - 10\mathbf{k})$ N and $\mathbf{F}_3 = (8\mathbf{i} + c\mathbf{j} - 2\mathbf{k})$ N are applied to a particle. The resultant force is three times the sum of $\mathbf{F}_1$ and $\mathbf{F}_2$.

Find the values of a, b and c.

18 A crow flies in the direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ with a speed of 30 miles per hour.

a Find its velocity vector.

b It takes off from point P. Find its position vector relative to P after 5 minutes.

19 Four points have coordinates $A(2, 4, 1)$, $B(k, 4, 2k)$, $C(k + 4, 2k + 4, 2k + 2)$ and $D(6, 2k + 4, 3)$.

a Show that $ABCD$ is a parallelogram for all values of k.

b When $k = 1$ find the angles of the parallelogram.

c Find the value of k for which $ABCD$ is a rectangle.

20 **a** Find the parametric equation of the line with vector equation $r = \begin{pmatrix} \frac{1}{2} \\ -2 \\ \frac{4}{3} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

b Determine whether the line intersects the x-axis.

c Find the angle the line makes with the x-axis.

21 Two lines have equations $l_1 : \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $l_2 : \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

a Find the acute angle between the lines.

b The two lines intersect at the point X. Find the coordinates of X.

c Show that the point $Y(9, -7, 3)$ lies on l_1.

d Point Z lies on l_2 such that XY is perpendicular to YZ. Find the area of the triangle XYZ.

22 Two particles move with constant velocity. Particle A starts from the point $(7, -3, 2)$ and moves with velocity $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ m s^{-1}. Particle B starts from the point $(1, 1, 26)$ and moves with velocity $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ m s^{-1}.

a Write down expression for the position vector of A and B after time t.

b **i** Show that the paths taken by the two particles intersect.

ii Find the coordinates of this point of intersection.

c Show further that the two particles do not collide.

23 In this question he base vectors **i** and **j** point East and North respectively.

A rowing boat, R, out at sea is first seen at 12:00 at the point with position vector $(-\mathbf{i} - 3\mathbf{j})$ km. At 12:40 it is seen at the point with position vector $(5\mathbf{i} - 5\mathbf{j})$ km.

a Calculate the bearing on which R is moving.

b Find an expression for the position vector of R at t hours after 12:00.

At 14:00 a speed boat leaves the origin travelling with speed $(p\mathbf{i} + q\mathbf{j})\,\text{km h}^{-1}$.

c Given that the speed boat intercepts R at 14:30, find the values of p and q.

24 The position vector of a particle at time t seconds is given by $\mathbf{r} = (4 + 3t)\mathbf{i} + (6 - t)\mathbf{j} + (2t - 7)\mathbf{k}$. The distance is measured in metres.

a Find the displacement of the particle from the starting point after 5 seconds.

b Find the speed of the particle.

c Determine whether the particle's path crosses the line connecting the points (3, 0, 1) and (1, 1, 5).

25 At time $t = 0$ two aircraft have position vectors $5\mathbf{j}$ and $7\mathbf{k}$. The first moves with velocity $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and the second with velocity $5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

a Write down the position vector of the first aircraft at time t.

b Show that at time t the distance, d, between the two aircraft is given by $d^2 = 44t^2 - 88t + 74$.

c Show that the two aircraft will not collide.

d Find the minimum distance between the two aircraft.

26 Let **a** and **b** be unit vectors and α the angle between them.

a Express $|\mathbf{a} - \mathbf{b}|$ and $|\mathbf{a} + \mathbf{b}|$ in terms of $\cos\alpha$.

b Hence find the value of α such that $|\mathbf{a} + \mathbf{b}| = 4|\mathbf{a} - \mathbf{b}|$.

27 Line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates (7, 2, 3). Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

28 Line l_1 has parametric equation $x = 2 + 4\lambda$, $y = -1 - 3\lambda$, $z = 3\lambda$.

Line l_2 is parallel to l_1 and passes through point $A(0, -1, 2)$.

a Write down the vector equation of l_2.

b Find the coordinates of the point B on l_1 such that AB is perpendicular to l_1.

c Hence find, to three significant figures, the shortest distance between the two lines.

29 Points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - 19\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\lambda\mathbf{i} + (\lambda + 2)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -6\mathbf{i} - 15\mathbf{j} + 7\mathbf{k}$.

a Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above

b find the angles of the triangle ABC

c find the area of the triangle ABC.

30 Let $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 5 \\ p \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

Find the value of p, given that $\mathbf{a} \times \mathbf{b}$ is parallel to $\mathbf{c}$.

31 Find the area of the triangle with vertices (2, 1, 2), (−1, 2, 2) and (0, 1, 5).

32 Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2q\mathbf{i} + \mathbf{j} + q\mathbf{k}$ find the values of scalars p and q such that $p\mathbf{a} + \mathbf{b}$ is parallel to vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

33 A kite has position vector at time t given by $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ m s^{-1}.

The top of a nearby tree has position vector $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ m.

Find the minimum distance between the kite and the treetop.

34 Two lines are given by $l_1 : \mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

a l_1 and l_2 intersect at P. Find the coordinates of P.

b Show that the point $Q(5, 2, 5)$ lies on l_2.

c Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

d Find the area of the triangle PQM.

35 Two lines have equations $l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

a Show that the point $P\left(\frac{5}{6}, \frac{19}{6}, \frac{9}{2}\right)$ lies on both lines.

b Find, in degrees, the acute angle between the two lines.

Point Q has coordinates (−1, 5, 10).

c Show that Q lies on l_2.

d Find the distance PQ.

e Hence find the shortest distance from Q to the line l_1.

36 Two lines with equations $l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ intersect at point P.

a Show that $Q(8, 2, 6)$ lies on l_2.

b R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R.

c Find a vector equation of a line through P which bisects the angle QPR.

37 **a** Line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates (7, 2, 3). Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

b Hence find the shortest distance from P to l.

c Q is a reflection of P in line l. Find the coordinates of Q.

38 Lines l_1 and l_2 have equations:

$l_1 : \mathbf{r} = (\mathbf{i} - 10\mathbf{j} + 12\mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

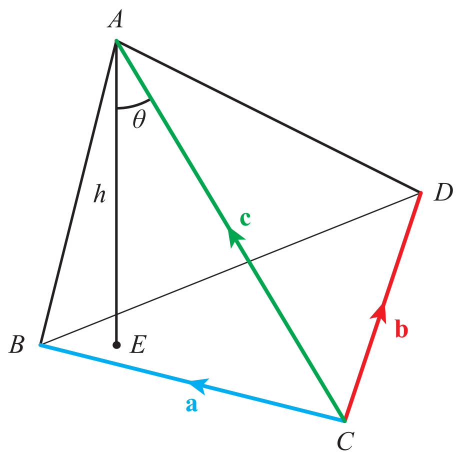
$l_2 : \mathbf{r} = (4\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{k})$

P is a point on l_1 and Q is a point on l_2 such that $\overrightarrow{PQ}$ is perpendicular to both lines.

a Show that $26\lambda + 7\mu = 64$ and find another equation for λ and μ.

b Hence find the shortest distance between the lines l_1 and l_2.

39 Consider the tetrahedron shown in the diagram and define vectors $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CD}$ and $\mathbf{c} = \overrightarrow{CA}$,.



a Write down an expression for the area of the base in terms of vectors **a** and **b** only.

b AE is the height of the tetrahedron, $|AE| = h$ and $\angle CAE = \theta$. Express h in terms of **c** and θ.

c Use the results of **a** and **b** to prove that the volume of the tetrahedron is given by $\left|\frac{1}{6}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right|$.

d Find the volume of the tetrahedron with vertices $A(0, 4, 0)$, $B(0, 6, 0)$, $C(1, 6, 1)$ and $D(3, -1, 2)$.

e Find the distance of the vertex A from the face BCD.

f Determine which of the vertices A and B is closer to its opposite face.

40 Port A is defined to be the origin of a set of coordinate axes and port B is located at the point (70,30), where distances are measured in kilometres. A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $\mathbf{r} = t\begin{pmatrix} 10 \\ 20 \end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

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3 Matrices

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to determine the order of a matrix
- how to add, subtract and multiply matrices
- some properties of matrix multiplication
- about zero and identity matrices
- how to calculate the determinant and inverse of $n \times n$ matrices with technology and 2×2 matrices by hand
- how to solve systems of linear equations with matrices
- how to find the eigenvalues and eigenvectors of 2×2 matrices
- how to diagonalize 2×2 matrices
- how to use diagonalization to find powers of 2×2 matrices.

CONCEPTS

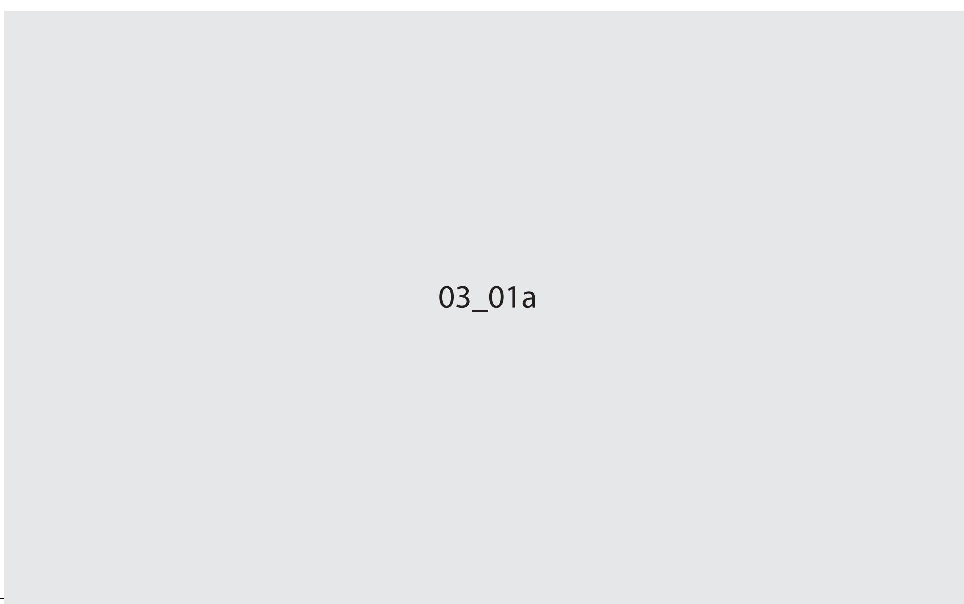
The following concepts will be addressed in this chapter:

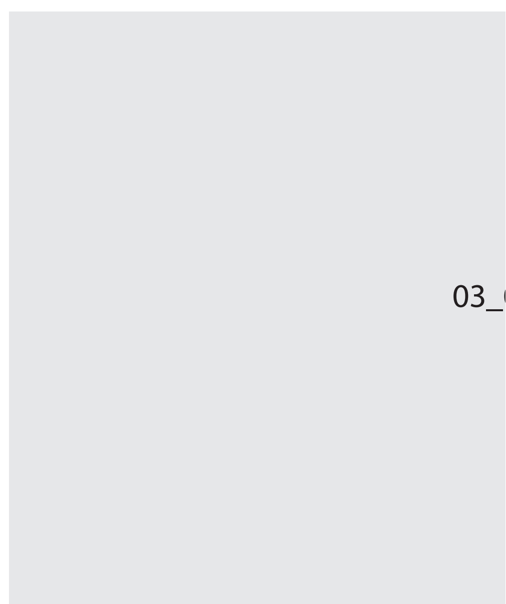
- Matrices allow us to organise data so that it can be manipulated and **relationships** be determined.

LEARNER PROFILE – to follow

Text to be inserted at proof stage.

Figure 3.1 Information can be displayed in different ways, such as, a football league table, pixels on a screen or allergen information on a menu





PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Calculate

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ b $\begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ c $4\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

2 Calculate $\left|\begin{pmatrix} 3 \\ -5 \end{pmatrix}\right|$

The idea of a column vector can be extended to a matrix by allowing two or more column vectors to be placed side by side in an array. Matrices provide an efficient way of storing and analysing large amounts of data and are used widely in many branches of mathematics from calculus to probability.

They form the basis of coding and encryption, computer programming, from graphics in a video game to a weather forecasting simulation, are used extensively in physics, for example in quantum mechanics.

Just as with numbers it is important to be able to perform operations such as addition, subtraction or multiplication with matrices and to be able to solve equations involving matrices.

Starter Activity

Look at the images in Figure 3.1. Discuss different ways to store and present the required information in each case.

Now look at this problem:

After 10 games of a league season, the results of four teams, Albion, City, Rovers and United are as follows:

	Won	Drawn	Lost
Albion	5	0	5
City	8	1	1
Rovers	2	3	5
United	2	7	1

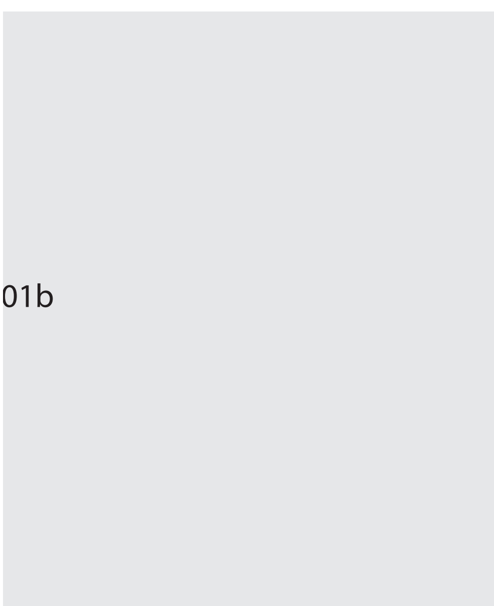
A new points system is operating in the league this season where a win is now worth 1 more point than before:

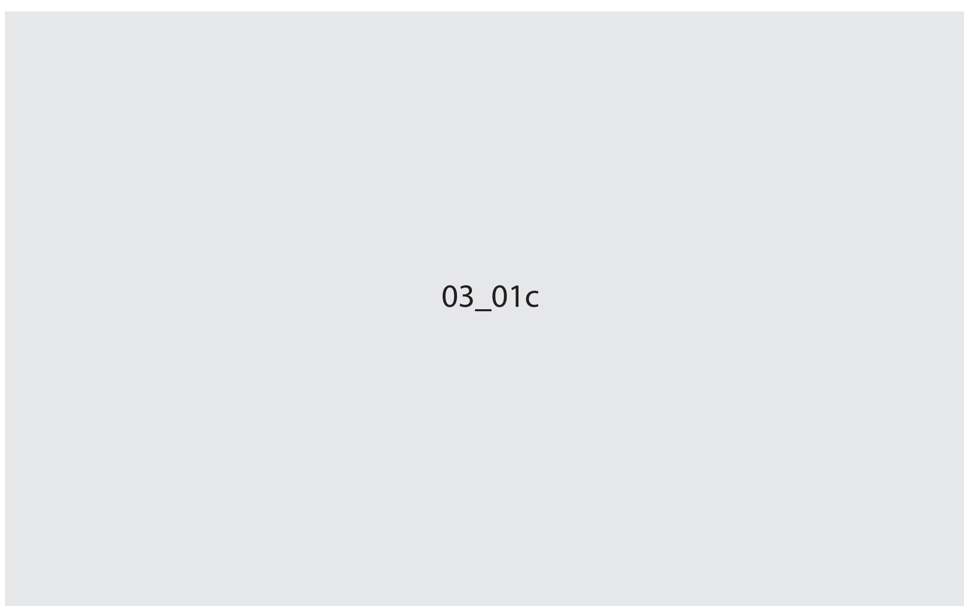
	New points	Old points
Win	3	2
Draw	1	1
Loss	0	0

a Find the number of points each team has under the new system.

b Find the number of points each team would have had under the old system.

c Has the change in points system affected the position of any of the teams?





3A Definition and arithmetic of matrices

Definition of a matrix

A **matrix** (plural **matrices**) is a rectangular array of **elements**, which may be numerical or algebraic. For example

$$\mathbf{A} = \begin{pmatrix} 2 & x & -3 \\ 0 & 4.5 & \pi \\ -1.8 & 1 & x^2 \end{pmatrix}$$

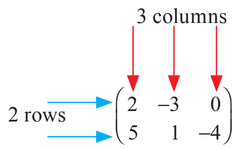
> **Tip**
>
> A matrix of order $n \times n$ (i.e. with the same number of rows as columns) is called a **square matrix**.

The number of rows and columns that a matrix has is called its **order**.

> **KEY POINT 3.1**
>
> A matrix with m rows and n columns has order $m \times n$.

WORKED EXAMPLE 3.1

State the order of the matrix $\begin{pmatrix} 2 & -3 & 0 \\ 5 & 1 & -4 \end{pmatrix}$



Order 2×3

> **Tip**
>
> Note that a column vector such as $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ can also be thought of as a matrix of order 2×1.

Algebra of matrices

Two matrices are equal when each of their corresponding elements is equal. Of course, for this to be possible, the matrices must first have the same order.

WORKED EXAMPLE 3.2

$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & x^2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} x+1 & 2 \\ -1 & y \end{pmatrix}$

Given that $\mathbf{A} = \mathbf{B}$, find the values of x and y.

$$\mathbf{A} = \mathbf{B}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & x^2 \end{pmatrix} = \begin{pmatrix} x+1 & 2 \\ -1 & y \end{pmatrix}$$

Equate the elements in the top left

$$3 = x + 1$$
$$x = 2$$

Equate the elements in the bottom right

$$x^2 = y$$
$$y = 4$$

If two matrices have the same order then one can be added to the other or one subtracted from the other.

KEY POINT 3.2

- Two matrices can be added/subtracted if they have the same order.
- To add/subtract matrices, add/subtract the corresponding elements.

WORKED EXAMPLE 3.3

$\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$

Find

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$

a $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$

Add the corresponding elements of each matrix

$= \begin{pmatrix} 6+1 & -1+4 \\ 2+3 & 3+(-7) \\ -3+(-2) & 5+8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 3 \\ 5 & -4 \\ -5 & 13 \end{pmatrix}$

b $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$

Subtract the corresponding element of matrix **B** from that of **A**

$= \begin{pmatrix} 6-1 & -1-4 \\ 2-3 & 3-(-7) \\ -3-(-2) & 5-8 \end{pmatrix}$

$= \begin{pmatrix} 5 & -5 \\ -1 & 10 \\ 5 & -3 \end{pmatrix}$

Make sure you also know how to use your GDC to add, subtract and multiply matrices.

Multiplication of a matrix by a scalar works in the same way as for multiplication of a vector by a scalar.

KEY POINT 3.3

To multiply a matrix by a scalar, multiply every element of the matrix by the scalar.

WORKED EXAMPLE 3.4

$$\mathbf{M} = \begin{pmatrix} -1 & 0.5 & 5 & 2.3 \\ 3 & 2 & 0 & -4 \end{pmatrix}$$

Find the matrix **A**, where **A** = 3**M**.

$$\mathbf{A} = 3\mathbf{M}$$
$$= 3\begin{pmatrix} -1 & 0.5 & 5 & 2.3 \\ 3 & 2 & 0 & -4 \end{pmatrix}$$

Multiply each element by 3

$$= \begin{pmatrix} 3(-1) & 3(0.5) & 3(5) & 3(2.3) \\ 3(3) & 3(2) & 3(0) & 3(-4) \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 1.5 & 15 & 6.9 \\ 9 & 6 & 0 & -12 \end{pmatrix}$$

Matrix multiplication

The product of two matrices **AB** is another matrix **C**. The process involves multiplying the elements of each row of **A** by the corresponding elements of each column of **B** and summing each time to find the entry of **C**.

WORKED EXAMPLE 3.5

Find the matrix **M** where

$$\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

The top left element is formed from the top row and the left column

$$\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 7 & -4 \end{pmatrix}$$

The middle right row is formed from the middle row and the right column

$$= \begin{pmatrix} 3\times 2 + (-1)\times 7 & 3\times 1 + (-1)(-4) \\ 2\times 2 + 4\times 7 & 2\times 1 + 4\times(-4) \\ (-5)\times 2 + 1\times 7 & (-5)\times 1 + 1\times(-4) \end{pmatrix}$$

And so on

$$= \begin{pmatrix} -1 & 7 \\ 32 & -14 \\ -3 & -20 \end{pmatrix}$$

Notice that the product of two matrices can only exist if the number of columns of the left-hand matrix is the same as the number of rows of the right-hand matrix.

KEY POINT 3.4

If the matrix **A** has order $m \times n$ and the matrix **B** has order $n \times p$, then the product **AB** has order $m \times p$.

WORKED EXAMPLE 3.6

Matrix **A** has order 3×5. Matrix **B** has order 4×3.

Find the order of the following products or explain why the product does not exist.

a **AB** b **BA**

A has 5 columns (3×5) and **B** has 4 rows (4×3) .... **a**

B has 3 columns (4×3) and **A** has 3 rows (3×5) .... **b** Order of **BA** is 4×5

Properties of matrix multiplication

It is clear from Worked Example 3.6 that in general the product **AB** is not the same as the product **BA** – in this case the product **AB** did not even exist!

This property of matrix arithmetic is different from usual arithmetic but other properties are the same.

KEY POINT 3.5

For the matrices **A**, **B** and **C**:
- $\mathbf{AB} \neq \mathbf{BA}$ (non-commutative)
- $\mathbf{A(BC)} = \mathbf{(AB)C}$ (associative)
- $\mathbf{A(B + C)} = \mathbf{AB} + \mathbf{AC}$ (distributive)

Be the Examiner

Expand $(\mathbf{A} + \mathbf{B})^1$, where **A** and **B** are both $n \times n$ matrices.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2$	$(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{B}^2 + \mathbf{AB} + \mathbf{BA}$	$(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$

Matrix multiplication is often useful in practical applications.

WORKED EXAMPLE 3.7

Two businesses, Any Tech and Bright Solutions, need to order varying quantities of three products P_1, P_2 and P_3 as given in the demand matrix, **D**:

$$\mathbf{D} = \begin{pmatrix} 80 & 55 & 75 \\ 70 & 60 & 60 \end{pmatrix} \begin{matrix} A \\ B \end{matrix}$$

The price per unit of these products from suppliers Supply All and Transit It are given in the cost matrix, **C**:

$$\mathbf{C} = \begin{pmatrix} £30 & £45 & £48 \\ £33 & £50 & £40 \end{pmatrix} \begin{matrix} S \\ T \end{matrix}$$

a Find the total cost matrix, **M**, that gives the overall cost to each business of ordering with each supplier.

b State which supplier each business should use to minimise cost.

You need to make sure that the quantities of P_1, P_2 and P_3 are multiplied by the price of P_1, P_2 and P_3 and added. This can be achieved by matrix multiplication if the rows of **C** are written as columns. We then have a 2×3 multiplied by a 3×2 matrix

Use your GDC to calculate the resulting 2×2 matrix

a $\mathbf{M} = \begin{matrix} A \\ B \end{matrix}\begin{pmatrix} 80 & 55 & 75 \\ 70 & 60 & 60 \end{pmatrix} \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix}\begin{pmatrix} 30 & 33 \\ 45 & 50 \\ 48 & 40 \end{pmatrix}$

The top left entry, for example, will be the total cost to business A of using supplier S

$= \begin{pmatrix} 8475 & 8390 \\ 7680 & 7710 \end{pmatrix} \begin{matrix} A \\ B \end{matrix}$

The top row gives the two potential costs to business A and the second row the two potential costs to business B

b Supplier T is cheaper for A (£8390 < £8475)
Supplier S is cheaper for B (£7680 < £7710)

Identity and zero matrices

The unique real number that has no effect when multiplied by another number is 1, that is, $1x = x$. The equivalent for matrices is the **identity matrix**.

KEY POINT 3.6

The identity matrix, **I**, is a square matrix with 1 as each element of the diagonal from top left to bottom right and 0 as every other element.

For example, the 2×2 identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the 3×3 identity is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The unique real number that has no effect when added to another number is 0, that is, $x + 0 = x$. The equivalent for matrices is the **zero matrix**.

KEY POINT 3.7

The zero matrix, **0**, is a matrix whose elements are all 0.

WORKED EXAMPLE 3.8

Given that $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$, find the matrix **B** where

$\mathbf{B} = \mathbf{A}^2 + 2\mathbf{I}$

A is a 2 × 2 matrix so $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{B} = \mathbf{A}^2 + 2\mathbf{I}$$

$$= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}^2 + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{A}^2$ means **A** multiplied by **A**

$$= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Finally, add the matrices

$$= \begin{pmatrix} 9 & 0 \\ -5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 \\ -5 & 6 \end{pmatrix}$$

Exercise 3A

In questions 1 to 3, use the method demonstrated in Worked Example 3.1 to find the order of the matrix.

1 a $\begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 5 \\ 0 & -6 & 3 \\ 2 & 4 & -1 \end{pmatrix}$

2 a $\begin{pmatrix} 7 & 0 \\ 0 & -1 \\ 9 & 3 \end{pmatrix}$

b $\begin{pmatrix} 3 & 8 & -5 & 0 \\ -4 & 2 & 1 & 3 \end{pmatrix}$

3 a $\begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix}$

b $\begin{pmatrix} -1 & 5 & 4 \end{pmatrix}$

In questions 4 and 5, use the method demonstrated in Worked Example 3.2 to find the values of x and y.

4 a $\begin{pmatrix} 1 & x \\ x-2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -y \\ 3 & 0 \end{pmatrix}$

b $\begin{pmatrix} x & -4 \\ y+3 & 2 \end{pmatrix} = \begin{pmatrix} 3-y & -4 \\ -1 & 2 \end{pmatrix}$

5 a $\begin{pmatrix} 2 & x+5 \\ x & y^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ x & x+3 \end{pmatrix}$

b $\begin{pmatrix} x^2 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} 9 & -y \\ -x & 1 \end{pmatrix}$

In questions 6 to 9, use the method demonstrated in Worked Example 3.3 to add/subtract the given matrices or state that this is not possible. Check your answers using your GDC.

6 a $\begin{pmatrix} 5 & -3 \\ 1 & 9 \end{pmatrix} + \begin{pmatrix} -4 & 10 \\ -2 & 7 \end{pmatrix}$

b $\begin{pmatrix} 5 & -3 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} -4 & 10 \\ -2 & 7 \end{pmatrix}$

7 a $\begin{pmatrix} 4 & 9 & 0 & -7 \\ -1 & 5 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 9 & -2 & -1 \\ 1 & 8 & -6 & 2 \end{pmatrix}$

b $\begin{pmatrix} 4 & 9 & 0 & -7 \\ -1 & 5 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 9 & -2 & -1 \\ 1 & 8 & -6 & 2 \end{pmatrix}$

8 a $\begin{pmatrix} 1 & 0 & 4 \\ 0 & -7 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 8 \\ -6 & 1 \\ 3 & -5 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 4 \\ 0 & -7 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 8 \\ -6 & 1 \\ 3 & -5 \end{pmatrix}$

9 a $\begin{pmatrix} 11 & -5 \end{pmatrix} - \begin{pmatrix} 3 & -2 \end{pmatrix}$

b $\begin{pmatrix} 11 & -5 \end{pmatrix} + \begin{pmatrix} 3 & -2 \end{pmatrix}$

In questions 10 to 12, use the method demonstrated in Worked Example 3.4 to multiply the scalar into the matrix. Check your answers using your GDC.

10 a $2\begin{pmatrix} 3 & 0 \\ 1 & -3 \\ -2 & 4 \end{pmatrix}$

b $5\begin{pmatrix} 3 & 0 \\ 1 & -3 \\ -2 & 4 \end{pmatrix}$

11 a $-4\begin{pmatrix} 0.5 & -1 & 3 \\ 2 & 0 & -1.2 \end{pmatrix}$

b $-3\begin{pmatrix} 0.5 & -1 & 3 \\ 2 & 0 & -1.2 \end{pmatrix}$

12 a $\frac{1}{5}\begin{pmatrix} -2 & 20 & 3 \\ 1 & -4 & -6.5 \\ 0.5 & 6 & 0 \end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix} -2 & 20 & 3 \\ 1 & -4 & -6.5 \\ 0.5 & 6 & 0 \end{pmatrix}$

In questions 13 to 17, use the method demonstrated in Worked Example 3.5 to find the matrix product or state that it does not exist. Check your answers using your GDC.

13 a $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 1 & -2 \end{pmatrix}$

b $\begin{pmatrix} 0 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$

14 a $\begin{pmatrix} 6 & 2 & 3 \\ 4 & -3 & 5 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ -2 & 2 & -1 \\ -3 & 4 & 0 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 5 \\ -2 & 2 & -1 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 6 & 2 & 3 \\ 4 & -3 & 5 \\ 1 & -2 & 0 \end{pmatrix}$

15 a $\begin{pmatrix} 4 & 1 \\ -2 & 5 \\ 3 & -1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \end{pmatrix}$

b $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 5 \\ 3 & -1 \\ 0 & -3 \end{pmatrix}$

16 a $\begin{pmatrix} 2 & 5 \\ -1 & -9 \end{pmatrix}\begin{pmatrix} 3 & 7 \end{pmatrix}$

b $\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -1 & -9 \end{pmatrix}$

17 a $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

b $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}$

In questions 18 to 20, use the method demonstrated in Worked Example 3.6 to find the order of the matrix product or state that it does not exist. In each case matrix **A** is 2×3, matrix **B** is 3×3 and matrix **C** is 4×2.

18 a **AB**

b **BA**

19 a **AC**

b **CA**

20 a **BC**

b **CB**

In questions 21 and 22, use the method demonstrated in Worked Example 3.8 to find the matrix **M.**

In each case $\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 3 & -2 & 5 \end{pmatrix}$.

21 a $\mathbf{M} = 2\mathbf{A} - 3\mathbf{I}$

b $\mathbf{M} = \mathbf{I} - \mathbf{A}$

22 a $\mathbf{A} - 4\mathbf{M} = 0$

b $\mathbf{M} + 2\mathbf{A} = 0$

23 Three teams, the Meteors, the Novas and the Orbits achieve results at home and away (Win, Draw and Lose) given by the matrices **H** and:

$$\mathbf{H} = \begin{pmatrix} 5 & 2 & 3 \\ 4 & 2 & 4 \\ 6 & 3 & 1 \end{pmatrix} \begin{matrix} M \\ N \\ O \end{matrix} \qquad \mathbf{A} = \begin{pmatrix} 4 & 1 & 5 \\ 4 & 0 & 6 \\ 5 & 2 & 3 \end{pmatrix} \begin{matrix} M \\ N \\ O \end{matrix}$$

Find the matrix that gives their overall results.

24 Last week's takings (in \$) in a book shop for IB Biology (B), IB Chemistry (C) and IB Physics (P) books at Standard Level (SL) and Higher Level (HL) are given in the following matrix:

$$\begin{matrix} B \\ C \\ P \end{matrix} \begin{pmatrix} 800 & 560 \\ 640 & 320 \\ 680 & 300 \end{pmatrix}$$

The manager projects an 8% increase in sales across all these books next week.

Find the matrix that gives next week's projected takings.

25 Find the values of x and y such that

$$\begin{pmatrix} 2x & 4 \\ 3 & y \end{pmatrix} + \begin{pmatrix} -y & -1 \\ 4 & 3x \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 7 & 13 \end{pmatrix}$$

26 Find the values of p and q such that

$$\begin{pmatrix} 1 & 2p & 0 \\ p & -2 & 3q \end{pmatrix} = \begin{pmatrix} 1 & 5-q & 0 \\ p & -2 & p-13 \end{pmatrix}$$

27 $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 5 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 5 \\ k & -k \end{pmatrix}$

Find, in terms of k

a $\mathbf{A} + 3\mathbf{B}$

b $2\mathbf{A} - \mathbf{B} + 4\mathbf{I}$

28 $\mathbf{A} = \begin{pmatrix} 1 & k \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -k & 2k \\ 1 & 3 \end{pmatrix}$

Find, in terms of k

a $\mathbf{A} - \mathbf{B}$

b $\mathbf{AB}$

29 $\mathbf{P} = \begin{pmatrix} 1 & k & -3 \\ 2 & 4 & k \\ 1 & 0 & 5 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 6 & 1 & 4 \\ -1 & k & 3 \\ 2 & 0 & k \end{pmatrix}$

Find, in terms of k

a $\mathbf{P} + \mathbf{Q}$

b $\mathbf{PQ}$

30 The following matrix shows the cost (in euros) of four products P_1, P_2, P_3 and P_4 in France and Germany.

$$\begin{matrix} F \\ G \end{matrix} \begin{pmatrix} 50 & 38 & 24 & 80 \\ 55 & 30 & 26 & 75 \end{pmatrix}$$

A business wants to buy 100 units of P_1, 50 units of P_2, 125 units of P_3 and 30 units of P_4.

a Write a matrix equation for the cost matrix, **C**.

b Hence find the cost of ordering from each country.

31 Find the values of x and y such that

$$\begin{pmatrix} x & y \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 5 & 0 \end{pmatrix}$$

32 Find the values of x and y such that

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & p \\ 3 & q \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -11 & 5 \end{pmatrix}$$

33 Two manufacturers, Engineer Right and Forge Well, need to produce varying quantities of four products P_1, P_2, P_3 and P_4 as given in the matrix **D**:

$$\mathbf{D} = \begin{pmatrix} 210 & 180 & 320 & 400 \\ 250 & 150 & 200 & 450 \end{pmatrix} \begin{matrix} E \\ F \end{matrix}$$

The price per unit (in \$) of producing these products at factories G and H is given in the matrix **C**:

$$\mathbf{C} = \begin{pmatrix} 55 & 80 & 48 & 20 \\ 50 & 85 & 40 & 22 \end{pmatrix} \begin{matrix} G \\ H \end{matrix}$$

a Find the total cost matrix **M** that gives the overall cost to each manufacturer of using each factory.

b State which factory each manufacturer should use to minimise cost.

34 The percentage of votes cast by males and females for the Red, Orange and Yellow parties at the last election is given by the following matrix.

$$\begin{matrix} R \\ O \\ Y \end{matrix} \begin{pmatrix} 28 & 34 \\ 42 & 31 \\ 30 & 35 \end{pmatrix}$$

The number of eligible voters in constituencies C_1 and C_2 at the next election is given by the following matrix.

$$\begin{matrix} C_1 \\ C_2 \end{matrix} \begin{pmatrix} 26124 & 30125 \\ 28987 & 29846 \end{pmatrix}$$

Assuming that the percentage of votes remains the same,

a calculate the 3×2 matrix that gives the projected number of votes for each party in these two constituencies at the next election

b find the projected total number of votes for the Yellow party in these two constituencies.

35 $\mathbf{A} = \begin{pmatrix} a & 3 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & b \\ 1 & 3 \end{pmatrix}$

Given that $\mathbf{A} + s\mathbf{B} = t\mathbf{I}$, find the values a, b, s and t.

36 $\boldsymbol{A} = \begin{pmatrix} 1 & a \\ 2 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b & 3 \\ -4 & -5 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix}$

Given that $\frac{1}{2}\boldsymbol{C} = p\boldsymbol{A} + q\boldsymbol{B}$, find the values of a, b, p and q.

37 $\boldsymbol{M} = \begin{pmatrix} 1 & 2a \\ -a & 3 \end{pmatrix}$ and $\boldsymbol{N} = \begin{pmatrix} 4 & b+1 \\ 3b & 1 \end{pmatrix}$

Given that $c\mathbf{M} + d\mathbf{N} = \mathbf{I}$, find the values of a, b, c and d.

38 Find the value of k so that the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} k & 6 \\ 4 & -1 \end{pmatrix}$ commute.

39 Given that the matrices $\begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix}$ and $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$ commute, find an expression for d in terms of c.

40 $\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$

a Calculate **AB** and **BC**.

b Show that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

3B Determinants and inverses

$n \times n$ matrices with technology

The **determinant** of a square matrix is a numerical value calculated from the elements of the matrix. It is similar to the magnitude of a vector, except that a determinant can be positive or negative.

You do not need to know how the determinant is calculated in general; you just need to be able to use your GDC to find the determinant.

You do need to be able to find the determinant of a 2×2 matrix without the GDC. You will see how to do this below.

Tip

In Section 4C you will see that the determinant of a matrix gives the scale factor of the enlargement determined by the matrix.

Tip

The notation $\mathbf{M}$ is usually used for the determinant of the matrix $\mathbf{M}$ but in keeping with the idea of a determinant being like the magnitude of a vector, the notation $|\mathbf{M}|$ is also used.

WORKED EXAMPLE 3.8

Find the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$.

$\det \mathbf{A} = 10$

Tip

Notice that a matrix must be square to have an inverse as otherwise the product $\mathbf{MM}^{-1}$ would not exist.

The reciprocal of the real number x (where $x \neq 0$) is the number x^{-1}, giving $xx^{-1} \neq 1$. An **inverse matrix** is the equivalent idea in matrices to that of a reciprocal in real numbers.

KEY POINT 3.8

The inverse of a square matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

WORKED EXAMPLE 3.9

Find the inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$.

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1.5 & -1 \\ -0.2 & -0.3 & 0.4 \\ -0.4 & -1.1 & 0.8 \end{pmatrix}$$

Finding the inverse of a matrix allows simple matrix equations to be solved. However, because matrix multiplication is not commutative, it is important to multiply the inverse on the correct side of the equation.

KEY POINT 3.9

- If $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$
- If $\mathbf{BA} = \mathbf{C}$, then $\mathbf{B} = \mathbf{CA}^{-1}$

The first result is proved here. The second result is proved similarly by multiplying $\mathbf{A}^{-1}$ on the right.

Proof 3.1

Prove that if $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$.

$$\mathbf{AB} = \mathbf{C}$$

Multiply through the equation on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{C}$$

Matrix multiplication is associative:

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$...

$$\mathbf{IB} = \mathbf{A}^{-1}\mathbf{C}$$

... and $\mathbf{IB} = \mathbf{B}$:

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

WORKED EXAMPLE 3.10

Matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$ and matrix $\mathbf{C} = \begin{pmatrix} 5 & -2 & 0 \\ -8 & 9 & 1 \end{pmatrix}$.

Given that $\mathbf{BA} = \mathbf{C}$, find the matrix $\mathbf{B}$.

Multiplying by $\mathbf{A}^{-1}$ on the right gives $\mathbf{B} = \mathbf{CA}^{-1}$:

$$\mathbf{BA} = \mathbf{C}$$
$$\mathbf{B} = \mathbf{CA}^{-1}$$

We found $\mathbf{A}^{-1}$ in Worked Example 3.11:

$$= \begin{pmatrix} 5 & -2 & 0 \\ -8 & 9 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1.5 & -1 \\ -0.2 & -0.3 & 0.4 \\ -0.4 & -1.1 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 5.4 & 8.1 & -5.8 \\ -10.2 & -15.8 & 12.4 \end{pmatrix}$$

CONCEPTS – RELATIONSHIPS

You can think of matrix B in Worked Example 3.10 as giving a relationship between matrices A and C - the two matrices are related through multiplication by B. This is a generalization of the idea of proportionality (when one quantity is a multiple of another). Another way to think of this type of a relationship is as a transformation: matrix B transforms A into C. We will apply this ideas to geometrical transformations in the next chapter.

Determinants and inverses of 2 × 2 matrices by hand

You need to be able to find the determinant and inverse of a 2 × 2 matrix by hand.

KEY POINT 3.10

The determinant of the 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$\det \mathbf{M} = ad - bc$

Tip

In the exam, you can always find numerical determinants by hand. However, it is useful to practice on numerical examples first before moving onto algebraic ones.

WORKED EXAMPLE 3.11

Find the determinant of the matrix $\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$.

$$\det \begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix} = (-3) \times 4 - (-5) \times 2$$

Use $\det \mathbf{M} = ad - bc$

$$= -12 - (-10)$$
$$= -2$$

If a matrix has a determinant of zero then the matrix is said to be **singular**.

WORKED EXAMPLE 3.12

Find the values of k for which the matrix $\begin{pmatrix} k & k-3 \\ 4 & k \end{pmatrix}$ is singular.

A singular matrix has $\det \mathbf{M} = 0$

$$\det \begin{pmatrix} k & 4 \\ k+3 & k \end{pmatrix} = 0$$

Use $\det \mathbf{M} = ad - bc$

$$k^2 - 4(k+3) = 0$$

Expand and solve with the GDC

$$k^2 - 4k - 12 = 0$$
$$k = 6 \text{ or } -2$$

Tip

You can see from the definition in Key Point 3.11 that a singular matrix does not have an inverse, because we cannot divide by zero.

You also need to be able to find the inverse of a 2 × 2 matrix by hand.

KEY POINT 3.11

The inverse of the 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det \mathbf{M} \neq 0$ is

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

WORKED EXAMPLE 3.13

Find the inverse of the matrix $\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$ from Worked Example 3.8.

Use $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

You already found that det $\mathbf{M} = -2$

$$\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -(-5) \\ -2 & -3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$$

You could either leave the factor of $-\frac{1}{2}$ out the front or multiply it into the matrix

$$= \begin{pmatrix} -2 & -2.5 \\ 1 & 1.5 \end{pmatrix}$$

Exercise 3B

In questions 1 to 6, use the method demonstrated in Worked Example 3.8 to find the determinant of each matrix with your GDC.

1 **a** $\begin{pmatrix} -2 & 1 & 3 \\ 6 & 4 & 1 \\ -3 & 2 & 5 \end{pmatrix}$

b $\begin{pmatrix} -2 & 3 & 1 \\ 4 & 1 & -1 \\ 5 & 0 & -2 \end{pmatrix}$

2 **a** $\begin{pmatrix} 3 & -18 & 2 \\ -1 & 13 & -3 \\ 1 & 8 & -4 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & 8 \\ 3 & 5 & 19 \\ 4 & 7 & 26 \end{pmatrix}$

3 **a** $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ -3 & 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & -2 \end{pmatrix}$

4 **a** $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 4 & 2 & 3 \end{pmatrix}$

5 **a** $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

b $\begin{pmatrix} 3 & -2 & 2 & 3 \\ 1 & 7 & -1 & 4 \\ 2 & 9 & -2 & 14 \\ 2 & 8 & -2 & 15 \end{pmatrix}$

6 **a** $\begin{pmatrix} 4 & -2 & 1 & 8 \\ 7 & 2 & 8 & 15 \\ 9 & -3 & 3 & 17 \\ 18 & -8 & -2 & 26 \end{pmatrix}$

b $\begin{pmatrix} -5 & 3 & 2 & 4 \\ 1 & -7 & 2 & -4 \\ 2 & 0 & 5 & -1 \\ 4 & 6 & 10 & 1 \end{pmatrix}$

In questions 7 to 12, use the method demonstrated in Worked Example 3.9 to find the inverse of each matrix with your GDC or state that it does not exist.

7 **a** $\begin{pmatrix} -2 & 1 & 3 \\ 6 & 4 & 1 \\ -3 & 2 & 5 \end{pmatrix}$

b $\begin{pmatrix} -2 & 3 & 1 \\ 4 & 1 & -1 \\ 5 & 0 & -2 \end{pmatrix}$

8 **a** $\begin{pmatrix} 3 & -18 & 2 \\ -1 & 13 & -3 \\ 1 & 8 & -4 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & 8 \\ 3 & 5 & 19 \\ 4 & 7 & 26 \end{pmatrix}$

9 **a** $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ -3 & 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & -2 \end{pmatrix}$

10 **a** $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 4 & 2 & 3 \end{pmatrix}$

11 **a** $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

b $\begin{pmatrix} 3 & -2 & 2 & 3 \\ 1 & 7 & -1 & 4 \\ 2 & 9 & -2 & 14 \\ 2 & 8 & -2 & 15 \end{pmatrix}$

12 **a** $\begin{pmatrix} 4 & -2 & 1 & 8 \\ 7 & 2 & 8 & 15 \\ 9 & -3 & 3 & 17 \\ 18 & -8 & -2 & 26 \end{pmatrix}$

b $\begin{pmatrix} -5 & 3 & 2 & 4 \\ 1 & -7 & 2 & -4 \\ 2 & 0 & 5 & -1 \\ 4 & 6 & 10 & 1 \end{pmatrix}$

In questions 13 to 16, use the method demonstrated in Worked Example 3.10 to find **B**. In these questions, $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 7 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 5 & -6 \\ 7 & -3 & 0 \\ 4 & 0 & -2 \end{pmatrix}$.

13 **a** $\mathbf{AB} = \begin{pmatrix} 7 & 9 & -7 \\ 19 & -21 & 11 \end{pmatrix}$

b $\mathbf{BA} = \begin{pmatrix} 6 & -8 \\ 5 & -2 \\ 1 & -2 \end{pmatrix}$

14 **a** $\mathbf{BA} = \begin{pmatrix} 25 & -22 \end{pmatrix}$

b $\mathbf{AB} = \begin{pmatrix} 1 \\ -9 \end{pmatrix}$

15 **a** $\mathbf{BC} = \begin{pmatrix} 13 & -4 & -2 \\ 19 & -15 & 8 \\ 6 & -24 & 26 \end{pmatrix}$

b $\mathbf{CB} = \begin{pmatrix} -12 & -15 & 5 \\ 14 & -2 & -10 \\ 4 & -6 & -4 \end{pmatrix}$

16 **a** $\mathbf{CB} = \begin{pmatrix} 9 \\ 5 \\ 6 \end{pmatrix}$

b $\mathbf{BC} = \begin{pmatrix} 5 & 1 & -4 \end{pmatrix}$

In questions 17 to 19, use the method demonstrated in Work Example 3.11 to find the determinant of each matrix.

17 **a** $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$

18 **a** $\begin{pmatrix} -6 & 2 \\ -5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -2 \\ -1 & 8 \end{pmatrix}$

19 **a** $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} -5 & 10 \\ -2 & 4 \end{pmatrix}$

In questions 20 to 22, use the method demonstrated in Worked Example 3.12 to find the value(s) of k for which each matrix is singular.

20 **a** $\begin{pmatrix} k & -3 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 5 \\ k & 2 \end{pmatrix}$

21 **a** $\begin{pmatrix} 6k & 3 \\ 2 & 4k \end{pmatrix}$ **b** $\begin{pmatrix} 9 & 2k \\ 2k & 4 \end{pmatrix}$

22 **a** $\begin{pmatrix} k-1 & k+5 \\ 2 & k \end{pmatrix}$ **b** $\begin{pmatrix} 2k & k-2 \\ k & 1 \end{pmatrix}$

In questions 23 to 25, use the method demonstrated in Worked Example 3.13 to find the inverse of each matrix (from questions 1 to 3) or to state that the inverse does not exist.

23 **a** $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$

24 **a** $\begin{pmatrix} -6 & 2 \\ -5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -2 \\ -1 & 8 \end{pmatrix}$

25 **a** $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} -5 & 10 \\ -2 & 4 \end{pmatrix}$

26 **a** Find the values of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2k & 1 \\ 4 & k \end{pmatrix}$ is singular.

b For all values of k apart from those in part **a**, find $\mathbf{A}^{-1}$ in terms of k.

27 a Show that the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3c \\ -c & 2 \end{pmatrix}$ has an inverse for all values of c.

b Find $\mathbf{A}^{-1}$ in terms of c.

28 $\mathbf{A} = \begin{pmatrix} 8 & 4 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 28 & 20 & 40 \\ -13 & -7 & -18 \end{pmatrix}$

Given that $\mathbf{AB} = \mathbf{C}$,

a find $\mathbf{A}^{-1}$

b hence find $\mathbf{B}$.

29 $\mathbf{B} = \begin{pmatrix} 2 & -1 & 6 \\ 1 & 3 & -2 \\ 7 & -4 & 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -9 & 19 & -31 \\ 25 & -10 & 38 \end{pmatrix}$

Given that $\mathbf{AB} = \mathbf{C}$,

a find $\mathbf{B}^{-1}$

b hence find $\mathbf{A}$.

30 A 6-digit number is written in a 3×2 matrix and encoded by multiplying by the matrix $\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{pmatrix}$.

The result is $\begin{pmatrix} 1 & -16 \\ 23 & 20 \\ 35 & 21 \end{pmatrix}$.

Find the original 3×2 matrix.

31 Given that $\boldsymbol{P} = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}$, find the matrix $\mathbf{Q}$ such that $\mathbf{P}^{-1}\mathbf{QP} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

32 Given that $\mathbf{A} = \begin{pmatrix} 5 & 1 & -4 \\ 3 & 0 & 2 \\ -1 & -1 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 2 & -2 \\ -3 & -1 & 2 \\ -5 & -1 & 2 \end{pmatrix}$ and that $\mathbf{ABC} = \mathbf{I}$, find the matrix $\mathbf{C}$.

33 For the matrix 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\det(k\mathbf{M}) = k^2 \det \mathbf{M}$.

34 $\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$

a Show that $\mathbf{M}^3 = 7\mathbf{M}^2 - 14\mathbf{M} + 8\mathbf{I}$.

b Hence deduce that $\mathbf{M}^{-1} = \frac{1}{8}(\mathbf{M}^2 - 7\mathbf{M} + 14\mathbf{I})$.

35 Prove that if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

36 Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Hence simplify $(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}$.

3C Solutions of systems of equations using the inverse matrix

You already know how to use the GDC to solve systems of two or three equations. You now also need to be able to set up these systems as a matrix equation and use your knowledge of inverse matrices to solve that equation.

Tip

Note that the system of equations will not have a unique solution if the matrix **A** is singular (since in this case $\mathbf{A}^{-1}$ does not exist).

KEY POINT 3.12

A system of simultaneous equations can be written as a matrix equation

$\mathbf{Ax} = \mathbf{b}$

where **A** contains the coefficients and x is a column vector of the variables.
If a solution exists, it is given by

$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

WORKED EXAMPLE 3.14

By first forming a matrix equation, solve the system of equations

$$\begin{cases} 2x - 7y = 5 \\ 3x - 8y = 4 \end{cases}$$

Set up the matrix equation $\mathbf{Ax} = \mathbf{b}$ where the entries of **A** are the coefficients of x and y

$$\begin{cases} 2x - 7y = 5 \\ 3x - 8y = 4 \end{cases}$$

$$\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

The solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Find $\mathbf{A}^{-1}$

$$= \frac{1}{5}\begin{pmatrix} -8 & 7 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -12 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -2.4 \\ -1.4 \end{pmatrix}$$

WORKED EXAMPLE 3.15

Hotel rooms in Alpha Hotel, Bravo Hotel and Charlie Hotel all offer nightly rates on Double (D), Superior (S) and Luxury (L) rooms as shown:

$$\begin{matrix} A \\ B \\ C \end{matrix} \begin{pmatrix} £120 & £175 & £225 \\ £125 & £200 & £275 \\ £100 & £150 & £250 \end{pmatrix}$$

A company wants to book several rooms for one night. It would cost £3 230 at Alpha, £3 575 at Bravo and £2 900 at Charlie.

Find the number of Double, Superior and Luxury rooms the company wants to book.

Let d be the number of double rooms, s be the number of superior rooms and l be the number of luxury rooms.
Then,

Set up a matrix equation of the form $\mathbf{A}x = \mathbf{b}$

$$\begin{pmatrix} 120 & 175 & 225 \\ 125 & 200 & 275 \\ 100 & 150 & 250 \end{pmatrix}\begin{pmatrix} d \\ s \\ l \end{pmatrix} = \begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

Solve the equation using $x = \mathbf{A}^{-1}\mathbf{b}$

$$\begin{pmatrix} d \\ s \\ l \end{pmatrix} = \begin{pmatrix} 120 & 175 & 225 \\ 125 & 200 & 275 \\ 100 & 150 & 250 \end{pmatrix}^{-1}\begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

Use the GDC to find the inverse of the 3 × 3 matrix

$$= -\frac{1}{900}\begin{pmatrix} -70 & 80 & -25 \\ 30 & -60 & 39 \\ 10 & 4 & -17 \end{pmatrix}\begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ 5 \\ 3 \end{pmatrix}$$

So, $d = 14$, $s = 5$, $l = 3$

Exercise 3C

As well as doing the questions in this exercise, revisit Exercise 12A of Mathematics: applications and interpretation SL and make sure you can set up and solve those systems of equations using matrices.

In questions 1 to 6, use the method demonstrated in Worked Example 3.14 to solve each system of equations by first forming a matrix equation.

1 **a** $\begin{cases} x - 4y = 11 \\ 3x + 5y = -1 \end{cases}$ **b** $\begin{cases} 2x + 3y = -5 \\ x + 5y = 1 \end{cases}$

2 **a** $\begin{cases} 6x - 4y = 7 \\ 9x - 6y = 2 \end{cases}$ **b** $\begin{cases} 3x + 9y = -4 \\ 4x + 12y = -1 \end{cases}$

3 **a** $\begin{cases} 7x - 3y = -1 \\ 4x - 2y = 3 \end{cases}$ **b** $\begin{cases} 3x + y = -2 \\ 8x + 4y = 5 \end{cases}$

4 **a** $\begin{cases} 2x+4y+z=8 \\ 3x+y-2z=7 \\ 5x-y+3z=9 \end{cases}$

b $\begin{cases} x+y+z=3 \\ x+2y+3z=-5 \\ 3x-2y+2z=4 \end{cases}$

5 **a** $\begin{cases} x+2y-z=2 \\ 2x+y=5 \\ x+z=4 \end{cases}$

b $\begin{cases} x-y=4 \\ y+z=1 \\ x-z=3 \end{cases}$

6 **a** $\begin{cases} 3x-y+z=17 \\ x+2y-z=8 \\ 2x-3y+2z=3 \end{cases}$

b $\begin{cases} x-2y-z=-2 \\ 2x+y-3z=9 \\ 5x+5y-8z=15 \end{cases}$

7 **a** Write the system of equations

$\begin{cases} 5x-4y=7 \\ 3x-2y=-1 \end{cases}$ in the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ column vectors.

b Find $\mathbf{A}^{-1}$.

c Hence solve the system of equations.

8 **a** Write the system of equations

$\begin{cases} x-y+z=1 \\ 5x+3y+2z=1 \\ 4y-3z=4 \end{cases}$ in the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ column vectors.

b Find $\mathbf{A}^{-1}$.

c Hence solve the system of equations.

9 A restaurant owner wants to buy a supply of raspberries (R) and strawberries (S). The cost matrix per kilogram (in euros) from suppliers X and Y is

$$\begin{matrix} X \\ Y \end{matrix} \begin{pmatrix} 8.5 & 5.5 \\ 8 & 6 \end{pmatrix}$$

€27 is spent with supplier x and €26.50 is spent with supplier Y.

Find the number of kilograms of raspberries and the number of kilograms of strawberries the restaurant owner buys.

10 Gill invested a total of \$50 000 in three different share funds, Allshare, Baserate and Chartwise. She invested \$10 000 more in Allshare than Chartwise.

After one year these funds had returned growth rates of 5.2%, 3.1% and 6.5% repectively. The total value of Gill's investment after one year was \$52 447.50.

a Set up a matrix equation of the form $\mathbf{Mx}=\mathbf{b}$ to represent this information.

b Hence find the amount Gill invested in each of the three share funds.

11 **a** Write the system of equations

$\begin{cases} kx+3y=-1 \\ (k-2)x+4y=1 \end{cases}$

in the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ column vectors.

b Hence find the value of k for which the equations do not have a solution.

c Assuming the value of k is not equal to that found in part b, solve the system of equations in terms of k.

12 **a** Write the system of equations

$\begin{cases} kx+5y=2 \\ 2x+(k-3)y=-1 \end{cases}$

in the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors.

b Hence find the values of k for which the equations do not have a solution.

c Assuming the value of k is not equal to either of those found in part **b**, solve the system of equations in terms of k.

3D Eigenvalues and eigenvectors

Tip

In this course the vector will always be two-dimensional so the matrix will be of order 2×2.

You will see geometrical applications of solutions to equations of this form in Section 4C and applications to probability in Section 8D.

Tip

In this course the characteristic equation will always be a quadratic.

Characteristic polynomial of 2×2 matrices

It is particularly useful to be able to solve equations of the form $\mathbf{Mv} = \lambda\mathbf{v}$, where $\mathbf{M}$ is a matrix, $\mathbf{v}$ is a column vector and λ is a scalar.

KEY POINT 3.13

If $\mathbf{Mv} = \lambda\mathbf{v}$ then λ is called an **eigenvalue** of the matrix $\mathbf{M}$ and the non-zero vector $\mathbf{v}$ its associated **eigenvector**.

To find the eigenvalues of a matrix you need to solve the **characteristic equation** of the matrix.

KEY POINT 3.14

The eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

Proof 3.2

Prove that the eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Rearrange the eigenvalue equation so that you have the zero matrix on the right

$$\mathbf{Mv} = \lambda\mathbf{v}$$

$$\mathbf{Mv} - \lambda\mathbf{v} = 0$$

Multiply $\lambda\mathbf{v}$ by the identity matrix so that $\mathbf{M} - \lambda\mathbf{I}$ is a square matrix

$$\mathbf{Mv} - \lambda\mathbf{Iv} = 0$$

$$(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = 0$$

If the matrix $\mathbf{M} - \lambda\mathbf{I}$ had an inverse then the equation $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = 0$ could be solved to give $\mathbf{v} = 0$, which is not permissible

Since $\mathbf{v} \neq 0$, the matrix $\mathbf{M} - \lambda\mathbf{I}$ is singular, so

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

WORKED EXAMPLE 3.16

Find the eigenvalues and associated eigenvectors of the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Solve the characteristic equation to find eigenvalues

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

$$\det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\det\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

You will always end up with a matrix of this form so you can start at this point in future

$$\det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = 0$$

Expand the determinant and solve for λ (by factorising or with your GDC)

$$(1-\lambda)(3-\lambda)-8=0$$
$$3-4\lambda+\lambda^2-8=0$$
$$\lambda^2-4\lambda-5=0$$
$$\lambda=5,\ -1$$

You now need to find the eigenvectors associated with these eigenvalues. To do this return to the definition of an eigenvector: $\mathbf{Mv}=\lambda\mathbf{v}$

When $\lambda = 5$,

$$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$$

Write this as two simultaneous equations

$$\begin{cases}x+4y=5x\\ 2x+3y=5y\end{cases}$$

Simplifying each, we see that they just give the same equation twice

$$\begin{cases}4x=4y\\ 2x=2y\end{cases}$$

So, you have a free choice of x or y. Any multiple of the vector $\begin{pmatrix}1\\ 1\end{pmatrix}$ would also be correct but this is the simplest version

$x=y$, so choosing $x=1$ gives the eigenvector $\mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$

When $\lambda = -1$,

Now repeat for the other eigenvalue

$$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{cases}x+4y=-x\\ 2x+3y=-y\end{cases}$$

Again you get the same equation twice, so you can choose x or y

$$\begin{cases}2x=-4y\\ 2x=-4y\end{cases}$$

The simplest choice this time is to let $y=1$ so that $x=-1$

$x=-2y$, so choosing $y=1$ gives the eigenvector $\mathbf{v}=\begin{pmatrix}-2\\ 1\end{pmatrix}$

(If you chose $x=1$ again you would get $y=-\frac{1}{2}$, which is still correct but not in such a simple form.)

Tip

The sum of the values on the diagonal running form top left of a matrix to bottom right will always be the same as the sum of the eigenvalues. This provides a useful way of checking whether the eigenvalues you get are likely to be correct.

Diagonalization of 2 × 2 matrices

Once you know the eigenvalues and eigenvectors of a matrix $\mathbf{M}$ you can find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{M}=\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, that is a matrix of the form $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$.

This process is called **diagonalization**.

Tip

In this course you only need to diagonalize matrices with two real distinct eigenvalues.

KEY POINT 3.15

Any matrix **M** with two real, distinct eigenvalues λ_1 and λ_2 can be expressed as $\mathbf{M} = \mathbf{PDP}^{-1}$

where **P** is a matrix whose columns are the eigenvectors of **M** and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Proof 3.3

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be the eigenvector corresponding to λ_1

and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be the eigenvector corresponding to λ_2

Use the definition of eigenvectors of **M**

Then,

$$\mathbf{M}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and

$$\mathbf{M}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Using the relationships above, the first column is $\lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the second $\lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\mathbf{MP} = \mathbf{M}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix}$$

Separate into the product of two matrices

$$= \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \mathbf{PD}$$

Multiply on the right by $\mathbf{P}^{-1}$

$$\mathbf{MPP}^{-1} = \mathbf{PDP}^{-1}$$

$$\mathbf{M} = \mathbf{PDP}^{-1}$$

WORKED EXAMPLE 3.17

Express the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ from Worked Example 3.14 in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

$\mathbf{M}$ has eigenvalues 5 and −1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

So,

Use $\mathbf{M} = \mathbf{PDP}^{-1}$, where the columns of $\mathbf{P}$ are the eigenvectors of $\mathbf{D}$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Applications to powers of 2 × 2 matrices

One of the main uses of diagonalization is in finding powers of a matrix. The key idea to note is that finding powers of a diagonal matrix is straightforward:

KEY POINT 3.16

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

This then leads to a result for the power of a matrix.

KEY POINT 3.17

If $\mathbf{M} = \mathbf{PDP}^{-1}$ for a diagonal matrix $\mathbf{D}$, then

$\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

Proof 3.4

$(\mathbf{PDP}^{-1})^n$ means $\mathbf{PDP}^{-1}$ multiplied together n times

$$\mathbf{M}^n = (\mathbf{PDP}^{-1})^n$$

$$= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\ldots(\mathbf{PDP}^{-1})$$

$$= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\ldots\mathbf{DP}^{-1}$$

Since matrix multiplication is associative we can group together each adjacent $\mathbf{PP}^{-1}$ to give $\mathbf{I}$

$$= \mathbf{PDIDIDI}\ldots\mathbf{DP}^{-1}$$

$$= \mathbf{PDDD}\ldots\mathbf{DP}^{-1}$$

$$= \mathbf{PD}^n\mathbf{P}^{-1}$$

WORKED EXAMPLE 3.18

For the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ from Worked Example 3.15,

a find $\mathbf{M}^n$ in terms of n

b hence find $\mathbf{M}^4$.

From Worked Example 3.17, $\mathbf{M}$ can be written in the form $\mathbf{M} = \mathbf{PDP}^{-1}$

a $$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Use $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

$$\mathbf{M}^n = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}^n\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Now use the fact that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Find the inverse of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ using Key Point 3.11 or with GDC

$$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Multiply together the first two matrices ...

$$= \begin{pmatrix} 5^n & -2(-1)^n \\ 5^n & (-1)^n \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

... and then multiply the result by the third matrix

$$= \begin{pmatrix} \frac{5^n + 2(-1)^n}{3} & \frac{2(5^n) - 2(-1)^n}{3} \\ \frac{5^n - (-1)^n}{3} & \frac{2(5^n) + (-1)^n}{3} \end{pmatrix}$$

Let $n = 4$ in the matrix from part **a**

b $$\mathbf{M}^4 = \begin{pmatrix} \frac{5^4 + 2(-1)^4}{3} & \frac{2(5^4) - 2(-1)^4}{3} \\ \frac{5^4 - (-1)^4}{3} & \frac{2(5^4) + (-1)^4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 209 & 416 \\ 208 & 417 \end{pmatrix}$$

Exercise 3D

In questions 1 to 3, use the method demonstrated in Worked Example 3.16 to find the eigenvalues and eigenvectors of each matrix.

1 **a** $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

2 **a** $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

3 **a** $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$ **b** $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

In questions 4 to 6, use the method demonstrated in Worked Example 3.17 to express each matrix (from questions 1 to 3) in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

4 **a** $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$

b $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

5 **a** $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$

b $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

6 **a** $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$

b $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

In questions 7 to 9, use the method demonstrated in Worked Example 3.18 to find $\mathbf{M}^n$ for each matrix (from questions 4 to 6).

7 **a** $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$

b $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

8 **a** $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$

b $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

9 **a** $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$

b $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

10 $\mathbf{M} = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$

a **i** Find the characteristic polynomial of $\mathbf{M}$.

ii Hence find the eigenvalues of $\mathbf{M}$.

b Find the eigenvectors of $\mathbf{M}$.

11 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 2 & 2 \\ 6 & -1 \end{pmatrix}$.

12 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$.

13 $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of matrix $\begin{pmatrix} 3 & 1 \\ 4 & a \end{pmatrix}$.

a Find the value of a.

b Find

i the eigenvalue associated with $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ii the remaining eigenvalue and associated eigenvector.

14 **a** Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}$

b Hence state a matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{CDC}^{-1}$.

15 $\mathbf{M} = \begin{pmatrix} 8 & 2 \\ -1 & 5 \end{pmatrix}$

a Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

b Hence find $\mathbf{M}^3$, clearly showing your working.

16 $\mathbf{M} = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}$

a Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

b Hence find $\mathbf{M}^n$.

17 $\mathbf{A} = \begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix}$

a Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

b Find $\mathbf{A}^n$ for integer n when

i n is odd

ii n is even.

18 Matrix **M** has eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ and eigenvalues 1 and –2.

a Find two possible matrices $\mathbf{M}_1$ and $\mathbf{M}_2$.

b Find the product $\mathbf{M}_1\mathbf{M}_2$ and comment on your answer.

19 $\mathbf{M} = \begin{pmatrix}0.25 & 0.5\\0.75 & 0.5\end{pmatrix}$

a Find the eigenvalues and corresponding eigenvectors of **M**.

b State a matrix **P** and a matrix **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where **D** is a diagonal matrix.

c Show that as n becomes large, $\mathbf{M}^n$ approaches

$$\begin{pmatrix}0.4 & 0.4\\0.6 & 0.6\end{pmatrix}$$

20 $\mathbf{M} = \begin{pmatrix}a & b\\a & b\end{pmatrix}$, where a and b are constants and $b \neq 0$.

a Find the eigenvalues of **M** in terms of a and b.

b Find corresponding eigenvectors of **M**.

c Express **M** in the form $\mathbf{M} = \mathbf{PDP}^{-1}$.

d Hence find $\mathbf{M}^n$.

e Given that $a = 0.6$ and $b = 0.4$, use your answer to part d) to find the matrix $\mathbf{M}^n$ approaches as n becomes very large.

Checklist

- You should be able to determine the order of a matrix.
  - A matrix with m rows and n columns has order $m \times n$.
- You should be able to add, subtract and multiply matrices.
  - Two matrices can be added/subtracted if they have the same order. To add/subtract matrices, add/subtract the corresponding elements.
  - To multiply a matrix by a scalar, multiply every element of the matrix by the scalar.
  - If the matrix **A** has order $m \times n$ and the matrix **B** has order $n \times p$, then the product **AB** has order $m \times p$.
- You should know some properties of matrix multiplication.
  - For the matrices **A**, **B** and **C**:
    — $\mathbf{AB} \neq \mathbf{BA}$ (non-commutative)
    — $\mathbf{A(BC)} = \mathbf{(AB)C}$ (associative)
    — $\mathbf{A(B + C)} = \mathbf{AB} + \mathbf{AC}$ (distributive)
- You should know about zero and identity matrices.
  - The identity matrix, **I**, is a square matrix with 1 as each element of the diagonal from top left to bottom right and 0 as every other element.
  - The zero matrix, 0, is a matrix whose elements are all 0.
- You should be able to calculate the determinant and inverse of $n \times n$ matrices with technology and 2×2 matrices by hand.
  - The inverse of a square matrix **M** is the matrix $\mathbf{M}^{-1}$ such that
    $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$
  - If $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$
  - If $\mathbf{BA} = \mathbf{C}$, then $\mathbf{B} = \mathbf{CA}^{-1}$
  - The determinant of the 2×2 matrix $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ is
    $\det \mathbf{M} = ad - bc$

- The inverse of the 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det \mathbf{M} \neq 0$ is
  $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- You should be able to solve systems of linear equations with matrices.
  - A system of simultaneous equations can be written as a matrix equation
    $\mathbf{Ax} = \mathbf{b}$
    — where $\mathbf{A}$ contains the coefficients and $\mathbf{x}$ is a column vector of the variables.
    — If a solution exists, it is given by
    $\mathrm{x} = \mathbf{A}^{-1}\mathbf{b}$

- You should be able to find the eigenvalues and eigenvectors of 2×2 matrices.

- If $\mathbf{Mv} = \lambda\mathbf{v}$ then λ is called an eigenvalue of the matrix $\mathbf{M}$ and the non-zero vector $\mathbf{v}$ its associated eigenvector.

- The eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation

  $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

- You should be able to diagonalize 2×2 matrices.
  - Any matrix $\mathbf{M}$ with two real, distinct eigenvalues λ_1 and λ_2 can be expressed as
    $\mathbf{M} = \mathbf{PDP}^{-1}$
    where $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{M}$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

- You should be able to use diagonalization to find powers of 2×2 matrices.
  - $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$
  - If $\mathbf{M} = \mathbf{PDP}^{-1}$ for a diagonal matrix $\mathbf{D}$, then $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

Mixed Practice

1 Matrix $\mathbf{R}$ gives the weekly revenue (in \$) from the sales of toy cats (C), dogs (D) and elephants (E) in shops X and Y of *Soft Toys 'r' Us*.

Matrix $\mathbf{F}$ gives the fixed costs of producing that number of toys and transporting them to the two stores.

$$\mathbf{R} = \begin{pmatrix} 110 & 85 \\ 154 & 102 \\ 130 & 175 \end{pmatrix} \begin{matrix} C \\ D \\ E \end{matrix} \qquad \mathbf{F} = \begin{pmatrix} 42 & 30 \\ 65 & 54 \\ 88 & 106 \end{pmatrix} \begin{matrix} C \\ D \\ E \end{matrix}$$

a Find the matrix $\mathbf{P}$, which give the weekly profit for each product at each store.

b **i** If the revenue of each product at each store increases by 5% and the costs increase by 2%, write an equation relating the new profit matrix $\mathbf{Q}$ to $\mathbf{R}$ and $\mathbf{F}$.

ii Hence find matrix $\mathbf{Q}$.

2 $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2k & -k \\ k & 1 \end{pmatrix}$

Find, in terms of k, the matrix $\mathbf{AB}$.

3 $A = \begin{pmatrix} 2k & 0 \\ k-1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$ where k is a constant.

a If **A** and **B** are commutative, find k.

b Show, by choosing matrices **C** and **D**, that matrix multiplication is not always commutative.

4 $A = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

Find $\mathbf{A}^{-1}$.

5 $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ -2 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ k & 4 \end{pmatrix}$

Given that **AC** = **B**, find **C**.

6 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 3 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

a Find $\mathbf{A}^{-1}$.

The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} 3 & k \\ 6 & 1 \end{pmatrix}$.

b Given that **CA** = **B**, find the matrix **C**.

7 Two 4-digit ID numbers are written in a 2 × 4 matrix and encoded by multiplying by the matrix

$$\begin{pmatrix} 9 & 18 & -13 & 3 \\ -3 & -6 & 4 & -1 \\ 7 & 12 & -15 & 2 \\ 1 & 1 & -13 & 1 \end{pmatrix}.$$

The result is $\begin{pmatrix} 68 & 127 & -130 & 22 \\ 119 & 218 & -240 & 38 \end{pmatrix}$.

Find the original ID numbers.

8 $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

a Show that $\mathbf{M}^2 = \mathbf{M}^{-3}$.

b Hence state the matrix $\mathbf{M}^3$.

c Use a matrix method to solve the equations

$$\begin{cases} 2x - y = 7 \\ 7x - 3y = 23 \end{cases}$$

9 Three sweet shops, *XS Sweets*, *Your Candy* and *Zorn's Confectionery* sell aniseed balls (A), bonbons (B) and caramels (C).

The price per kilogram (£) of each type of sweet at the three shops are as follows:

$$\begin{matrix} X \\ Y \\ Z \end{matrix} \begin{pmatrix} 1.00 & 1.80 & 2.50 \\ 1.10 & 2.00 & 2.10 \\ 1.20 & 1.70 & 2.40 \end{pmatrix}$$

Jack wants to order a certain number of kilograms of each sweet. He works out that his order would cost him £24.90 at *XS Sweets*, £22.90 at *Your Candy* and £24.30 at *Zorn's Confectioner.*

How many kilograms of each sweet does he want to order?

10 $\mathbf{M} = \begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$

a **i** Find the characteristic polynomial of **M**.

ii Hence find the eigenvalues of **M**.

b Find the eigenvectors of **M**.

11 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$.

12 $\mathbf{A} = \begin{pmatrix} a & -1 \\ 2 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 3 & b \\ 0 & -4 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 1 & 4 \\ 2 & c \end{pmatrix}$

Given that $\mathbf{AB} + k\mathbf{C} = \mathbf{I}$, find a, b, c and k.

13 $\mathbf{M} = \begin{pmatrix} 3 & k \\ 5 & 8 \end{pmatrix}$

a **i** Find the value of k for which **M** is singular.

ii Given that **M** is non-singular, find $\mathbf{M}^{-1}$ in terms of k.

b When $k = 7$, use $\mathbf{M}^{-1}$ to solve the simultaneous equations

$$\begin{cases} 3x + 7y = -1 \\ 5x + 8y = 13 \end{cases}$$

14 **a** If $\mathbf{ABC} = \mathbf{I}$, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & -1 \\ -3 & 4 \end{pmatrix}$

Find **B**.

15 $\mathbf{M} = \begin{pmatrix} x & 2 \\ y & 0 \end{pmatrix}$

Given $\mathbf{M}^{-1}$ exists and $\mathbf{M} + \mathbf{M}^{-1} = \mathbf{I}$, find x and y.

16 A matrix **M** has eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, with corresponding eigenvalues 2 and 3.

Find the matrix M.

17 $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$

a Find matrices **P** and **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where **D** is a diagonal matrix.

b Hence find $\mathbf{M}^n$.

18 $\mathbf{M} = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$, where $0 < a < 1$ and $0 < b < 1$.

a **i** Show that one of the eigenvalues of **M** is $\lambda = 1$ and find the other.

ii Find the corresponding eigenvectors of **M**.

b State matrices **P** and **P**, where **D** is a diagonal matrix, such that $\mathbf{M} = \mathbf{PDP}^{-1}$.

c Hence find $\mathbf{M}^n$.

4 Geometry and Trigonometry

ESSENTIAL UNDERSTANDINGS

- Trigonometry allows us to quantify the physical world.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn...

- about different units for measuring angles, called radians
- how to find the length of an arc of a circle
- how to find the area of a sector of a circle
- how to define the sine and cosine functions in terms of the unit circle
- how to define the tangent function
- about the ambiguous case of the sine rule
- about the Pythagorean identity $\cos^2 \theta + \sin^2 \theta \equiv 1$
- how to sketch the graphs of trigonometric functions
- how to solve trigonometric equations graphically
- how to use matrices to represent geometrical transformations
- how to combine transformations
- how to find the area of an image after a transformation.

CONCEPTS

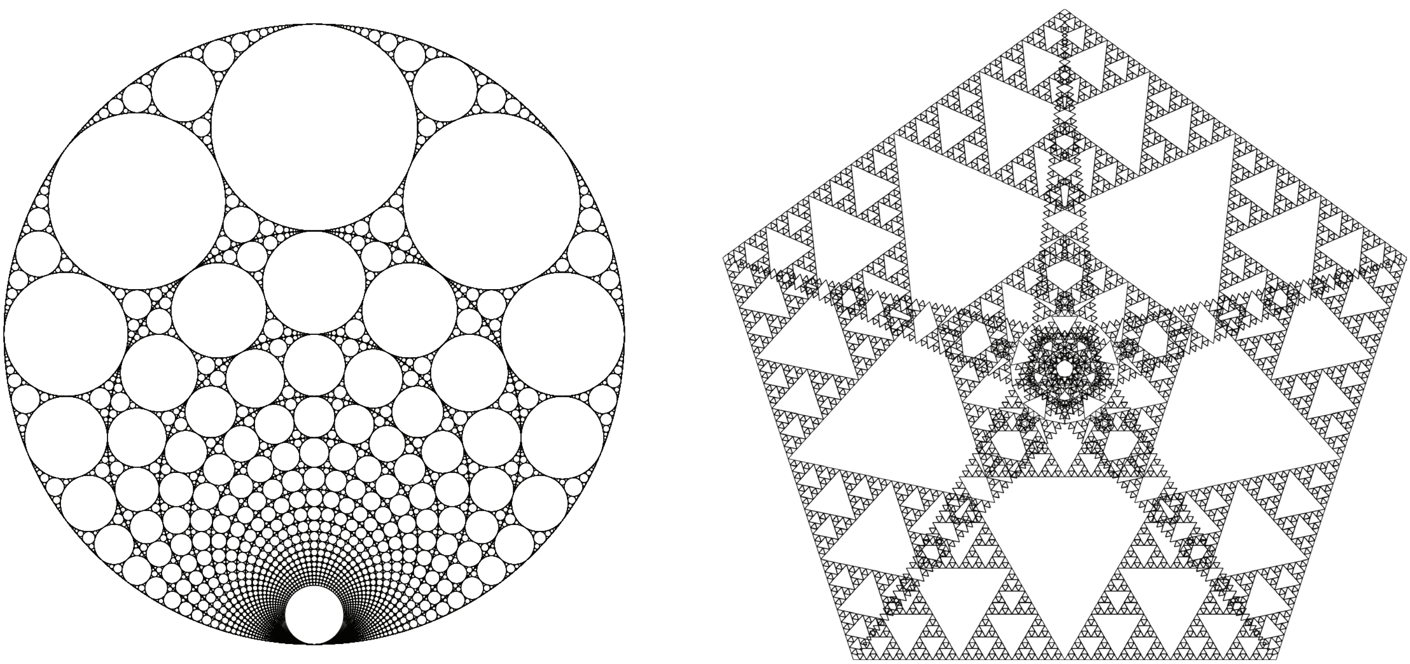
The following concepts will be addressed in this chapter:

- **Equivalent** measurement systems, such as degrees and radians, can be used for angles to facilitate our ease of calculation.
- Different **representations** of the values of trigonometric relationships, such as exact or **approximate**, may not be **equivalent** to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically **represent** the periodic or symmetric nature of their values.

LEARNER PROFILE – to follow

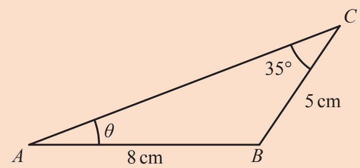
Text to be inserted at proof stage.

Figure 4.1 Trigonometric functions and the repeated application of transformation matrices can combine to make detailed geometric designs.



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:



1 Find the angle θ.

2 Use technology to solve the equation $2^x = x^3 - 4x$.

3 Given that $A = \begin{pmatrix} 2 & 1 \\ -3 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, find

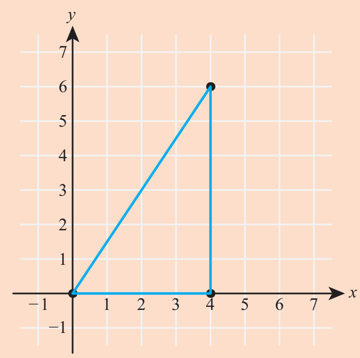
a $\mathbf{AC}$ **b** $\mathbf{BA}$ **c** $\mathbf{B}^{-1}$

4 Draw the image of the triangle in the diagram after the following transformations:

a reflection in the line $y = x$

b rotation 90° clockwise around the origin

c translation with vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.



Up until now you have used trigonometric functions (sine, cosine and tangent) to find side lengths and angles in triangles, measuring angles in degrees. It turns out that a different measure of angle, the radian, is more useful when modelling other real-life situations with trigonometric functions.

Trigonometric functions can be combined with matrices to calculate coordinates of points after geometrical transformations such as rotations and reflections. This is used, for example, to generate patterns in graphic design.

Starter Activity

Look at the pictures in Figure 4.1. In small groups, identify which transformations were used to generate these pictures.

Now look at this problem:

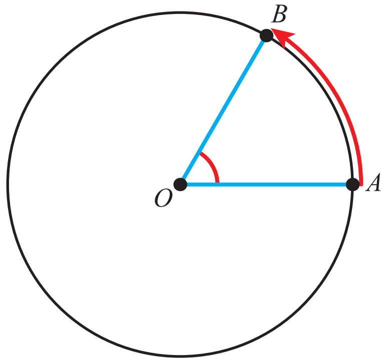
1 How many triangles can you draw given the following information: $AB = 15\text{cm}$, $BC = 10\text{cm}$, $A\hat{C}B = 40°$?

2 What if the information is as given above except that $BC = 18\text{cm}$?

4A Radian measure

Although degrees are the units you are familiar with for measuring angles, they are not always the best unit to use. Instead, **radians** are far more useful in many branches of mathematics.

Radians relate the size of an angle at the centre of the unit circle (a circle with radius 1) to the distance a point moves round the circumference of that circle:



The length of the arc AB is equal to the size of angle AOB in radians.

Since the circumference of the unit circle is $2\pi \times 1 = 2\pi$, there are 2π radians in one full turn.

Tip

Radians are often given as multiples of π, but can also be given as decimals.

KEY POINT 4.1

$360° = 2\pi$ radians

WORKED EXAMPLE 4.1

a Convert 75° to radians.

b Convert 1.5 radians to degrees.

$360° = 2\pi$ radians, so $1° = \frac{2\pi}{360}$ radians

a $75° = \frac{2\pi}{360} \times 75$

$= \frac{75\pi}{180}$

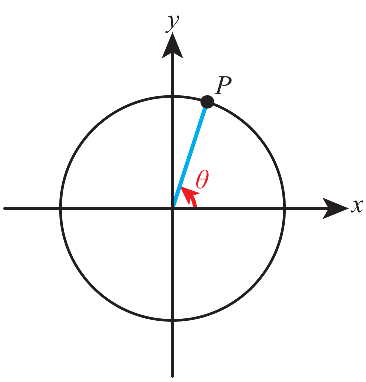
$= \frac{5\pi}{12}$ radians

2π radians $= 360°$, so 1 radian $= \left(\frac{360}{2\pi}\right)°$

b 1.5 radians $= \frac{360}{2\pi} \times 1.5$

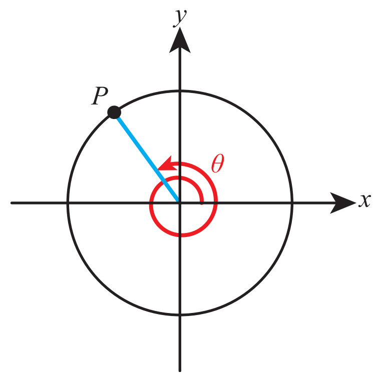
$= 85.9°$

Using the unit circle, we can define any positive angle as being between the positive x-axis and the radius from a point P as P moves anti-clockwise around the circle:

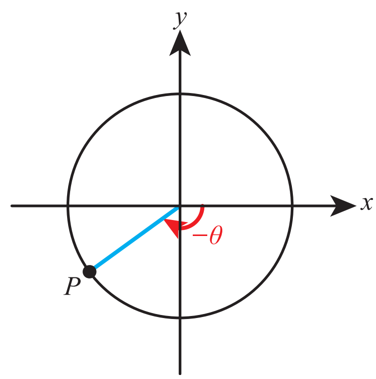


During the French Revolution there was a move towards decimalization, including the introduction of a new unit called the gradian (often abbreviated to grad), which split right angles into 100 subdivisions. It is still used in some areas of surveying today.

If the angle is greater than 2π, P just goes around the circle again:



If the angle is negative, P moves clockwise from the positive x-axis:



WORKED EXAMPLE 4.2

Mark on the unit circle the points corresponding to these angles:

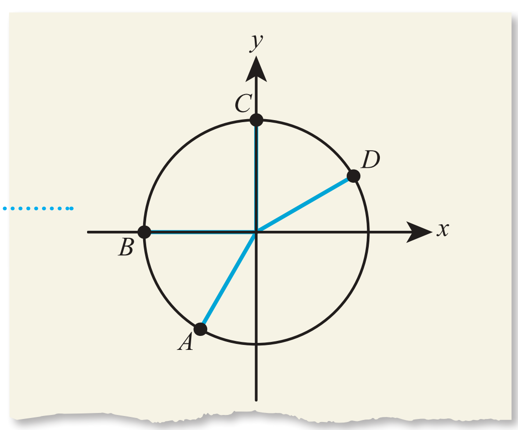
A $\frac{4\pi}{3}$ **B** -3π **C** $\frac{5\pi}{2}$ **D** $\frac{25\pi}{6}$

A: $\frac{4\pi}{3} = \frac{2}{3} \times 2\pi$ so it is $\frac{2}{3}$ of a whole turn

B: $-3\pi = -2\pi - \pi$ so it is a whole turn 'backwards' (clockwise) followed by half a turn in the same negative direction

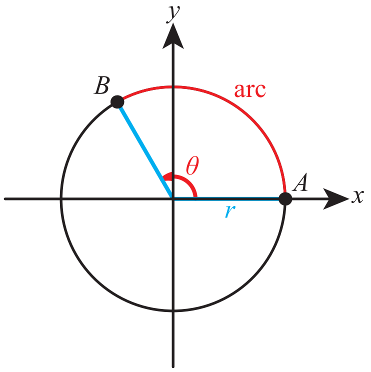
C: $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ so it is a whole turn followed by $\frac{1}{4}$ of a turn

D: $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$ so it is two whole turns followed by $\frac{1}{12}$ of a turn



Length of an arc

The arc AB **subtends** an angle θ at the centre of the circle.



Since the ratio of the arc length, s, to the circumference will be the same as the ratio of θ to 2π radians, this gives us:

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

Rearranging gives the formula for arc length when θ is measured in radians.

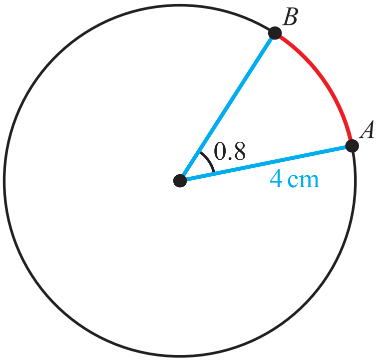
KEY POINT 4.2

The length of an arc is $s = r\theta$

where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

WORKED EXAMPLE 4.3

Find the length of the arc AB in the circle shown.

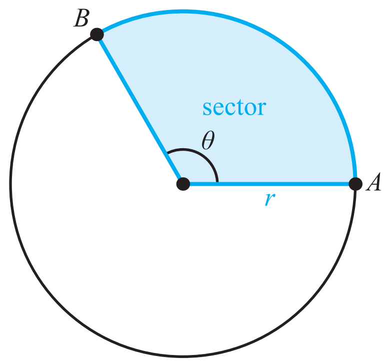


Use $s = r\theta$ with θ in radians

$$s = r\theta$$
$$= 4 \times 0.8$$
$$= 3.2\,\text{cm}$$

Area of a sector

By a very similar argument to that above, we can obtain a formula for the area of a sector.



Since the ratio of the area of a sector, A, to the area of the circle will be the same as the ratio of θ to 2π radians, this gives us:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

Rearranging gives the formula for the area of a sector when θ is measured in radians.

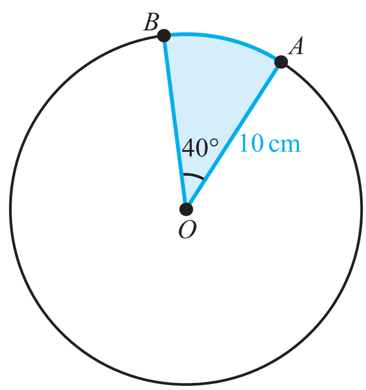
KEY POINT 4.3

The area of a sector is $A = \frac{1}{2}r^2\theta$

where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

WORKED EXAMPLE 4.4

Find the area of the sector AOB in the circle shown.



First convert 40° to radians

$$40° = \frac{2\pi}{360} \times 40$$

$$= \frac{2\pi}{9} \text{ radians}$$

Then use $A = \frac{1}{2}r^2\theta$

$$A = \frac{1}{2} \times 10^2 \times \frac{2\pi}{9}$$

$$= 34.9 \text{ cm}^2$$

KEY CONCEPTS – EQUIVALENCE

You have seen various quantities that can be measured in different units. For example, lengths can be measured in feet or metres and temperature in degrees Fahrenheit or Celsius. There is no mathematical reason to prefer one over the other for them because the units that we use to measure in do not change any formulae used; they are **equivalent**. You might have expected it to be the same for this new unit of angle measurement, however, as seen in Worked Example 18.4, in this case the units used actually do change the formulae. This has some very important consequences when it comes to differentiating trigonometric functions, as you will see in Chapter 20.

You met the sine rule, cosine rule, and area of a triangle formula in Section 5B of the Mathematics: applications and interpretation SL book.

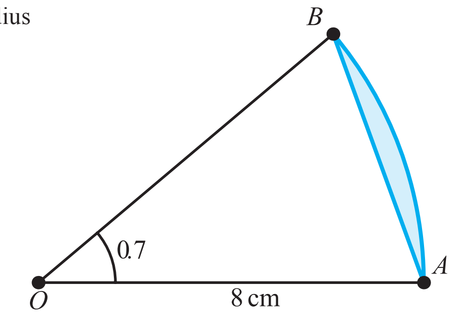
Tip

If you are working with $\sin\theta$, $\cos\theta$ or $\tan\theta$ with θ in radians, you must set your calculator to radian (not degree) mode.

You will often need to combine the results for arc length and area of sector with the results you know for triangles.

WORKED EXAMPLE 4.5

The diagram shows a sector of a circle with radius 8 cm and angle 0.7 radians.



For the shaded region, find:

a the perimeter
b the area.

Use $s = r\theta$ to find the arc length

a $s = 8 \times 0.7$

$= 5.6$

Use the cosine rule to find the length of the chord AB. Remember to make sure your calculator is in radian mode

By cosine rule,

$AB^2 = 8^2 + 8^2 - 2 \times 8 \times 8\cos 0.7$

$= 30.1002$

So, $AB = 5.4864$

The perimeter of the shaded region is the length of the arc plus the length of the chord

Hence, $p = 5.6 + 5.49$

$= 11.1\text{ cm}$

Use $A = \frac{1}{2}r^2\theta$ to find the area of the sector

b Area of sector $= \frac{1}{2} \times 8^2 \times 0.7$

$= 22.4$

Use $A = \frac{1}{2}ab\sin C$ with $a = b = r$ and $C = \theta$ to find the area of the triangle

Area of triangle $= \frac{1}{2} \times 8^2 \sin 0.7$

$= 20.615$

The area of the shaded region is the area of the sector minus the area of the triangle

$A = 22.4 - 20.6$

$= 1.79\text{ cm}^2$

Exercise 4A

For questions 1 to 12, use the method demonstrated in Worked Example 4.1 to convert between degrees and radians.

Convert to radians, giving your answers in terms of π:

1 **a** $60°$ **b** $45°$

2 **a** $150°$ **b** $120°$

3 **a** $90°$ **b** $270°$

Convert to radians, giving your answer to three significant figures:

4 **a** $28°$ **b** $36°$

5 **a** $67°$ **b** $78°$

6 **a** $196°$ **b** $236°$

Convert to degrees, giving your answers to one decimal place where appropriate:

7 **a** 0.62 radians **b** 0.83 radians

8 **a** 1.26 radians **b** 1.35 radians

9 **a** 4.61 radians **b** 5.24 radians

10 **a** $\frac{\pi}{5}$ radians **b** $\frac{\pi}{8}$ radians

11 **a** $\frac{7\pi}{12}$ radians **b** $\frac{4\pi}{15}$ radians

12 **a** $\frac{7\pi}{3}$ radians **b** $\frac{11\pi}{6}$ radians

For questions 13 to 18, use the method demonstrated in Worked Example 4.2 to mark on the unit circle the points corresponding to these angles:

13 **a** $\frac{2\pi}{3}$ **b** $\frac{3\pi}{4}$

14 **a** $\frac{5\pi}{6}$ **b** $\frac{7\pi}{4}$

15 **a** $-\frac{\pi}{2}$ **b** $-\frac{3\pi}{2}$

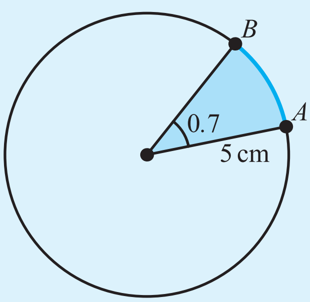
16 **a** $-\frac{\pi}{3}$ **b** $-\frac{\pi}{4}$

17 **a** $\frac{8\pi}{3}$ **b** $\frac{11\pi}{4}$

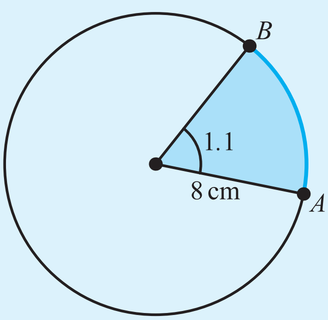
18 **a** $\frac{11\pi}{2}$ **b** $\frac{19\pi}{2}$

For questions 19 to 21, use the methods demonstrated in Worked Examples 4.3 and 4.4 to find the length of the arc *AB* that subtends the given angle (in radians), and the area of the corresponding sector.

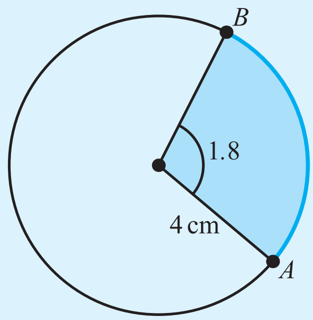
19 **a**



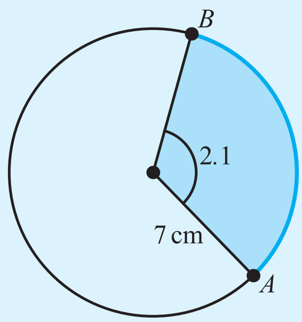
b



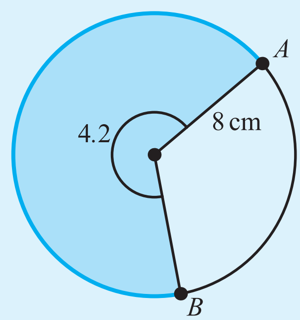
20 **a**



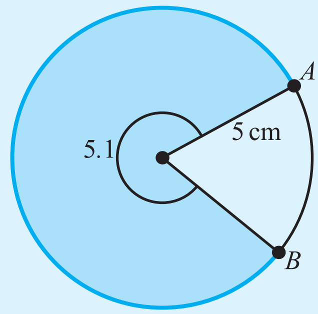
b



21 **a**

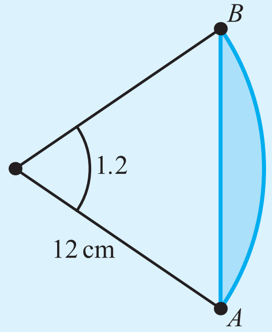


b

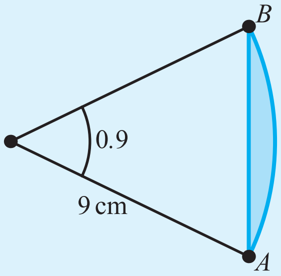


For questions 23 to 24, use the method demonstrated in Worked Example 4.5 to find the area and the perimeter of the shaded region.

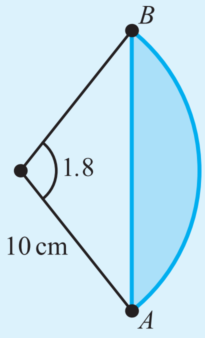
22 **a**



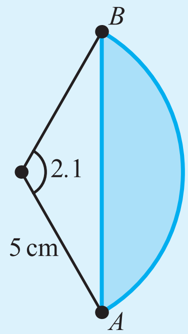
b



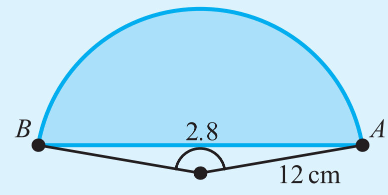
23 **a**



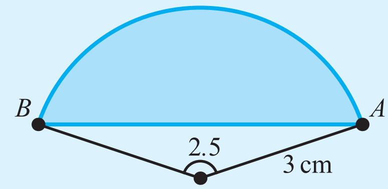
b



24 **a**

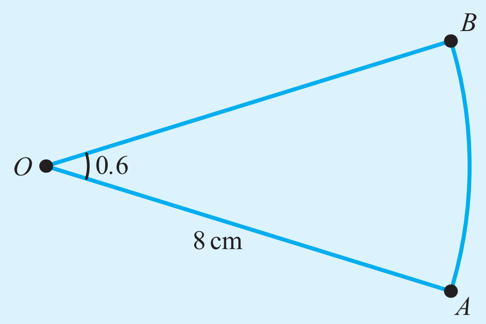


b



25 The diagram shows a sector *AOB* of a circle of radius 8 cm. The angle of the centre of the sector is 0.6 radians.

Find the perimeter and the area of the sector.



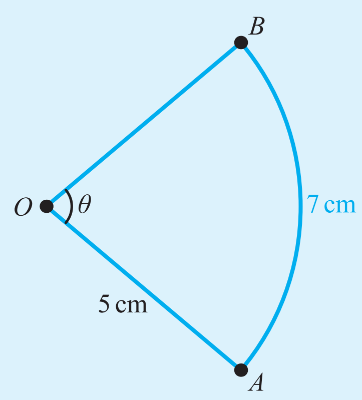
26 A circle has centre *O* and radius 6.2 cm. Points *A* and *B* lie on the circumference of the circle so that the arc *AB* subtends an angle of 2.5 radians at the centre of the circle. Find the perimeter and the area of the sector *AOB*.

27 An arc of a circle has length 12.3 cm and subtends an angle of 1.2 radians at the centre of the circle. Find the radius of the circle.

28 The diagram shows a sector of a circle of radius 5 cm. The length of the arc *AB* is 7 cm.

a Find the value of θ.

b Find the area of the sector.



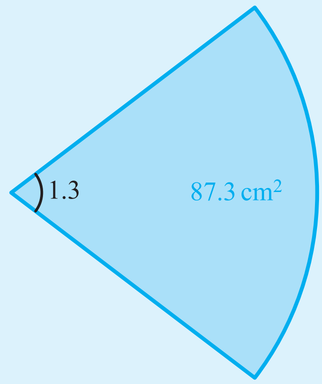
29 A circle with centre O has radius 23 cm. Arc AB subtends angle θ radians at the centre of the circle. Given that the area of the sector AOB is 185 cm^2, find the value of θ.

30 A sector of a circle has area 326 cm^2 and an angle at the centre of 2.7 radians. Find the radius of the circle.

31 The diagram shows a sector of a circle. The area of the sector is 87.3 cm^2.

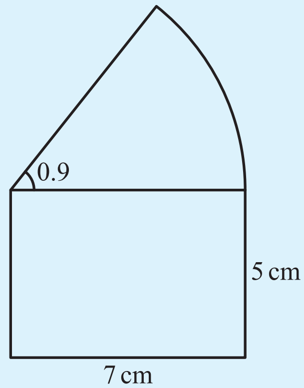
a Find the radius of the circle.

b Find the perimeter of the sector.



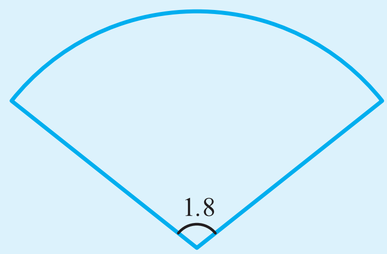
32 The figure shown in the diagram consists of a rectangle and a sector of a circle.

Calculate the area and the perimeter of the figure.



33 The diagram shows a sector of a circle.

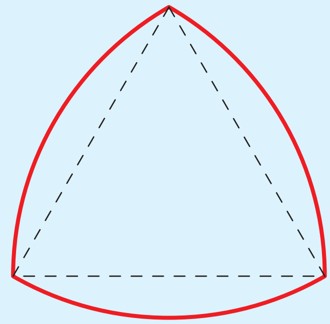
The perimeter of the sector is 26 cm. Find the radius of the circle.



34 A sector of a circle has area 18 cm^2 and perimeter 30 cm. Find the possible values of the radius of the circle.

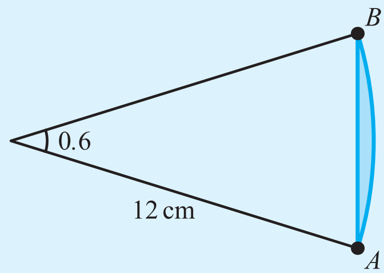
35 The diagram shows an equilateral triangle and three arcs of circles with centres that are the vertices of the triangle.

The length of the sides of the triangle are 12 cm. Find the perimeter of the figure.



36 A circle has centre O and radius 8 cm. Chord PQ subtends angle 0.9 radians at the centre. Find the difference between the length of the arc PQ and the length of the chord PQ.

37 The arc AB of a circle of radius 12 cm subtends an angle of 0.6 radians at the centre. Find the perimeter and the area of the shaded region.

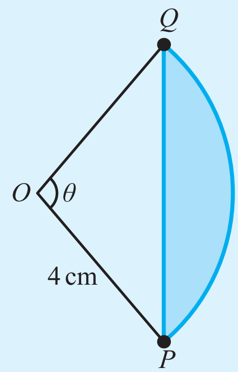


38 A circle has centre O and radius 4 cm. Chord PQ subtends angle θ radians at the centre. The area of the shaded region is 6 cm^2.

a Show that $\theta - \sin\theta = 0.75$.

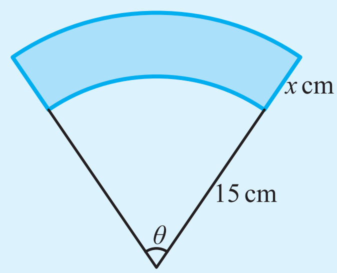
b Find the value of θ.

c Find the perimeter of the shaded region.



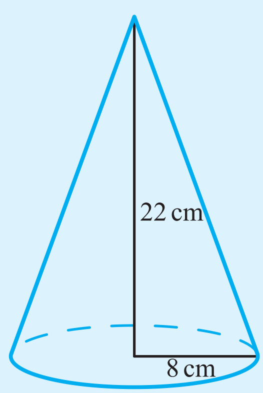
39 The diagram shows two circular sectors with angle 1.2 radians at the centre. The radius of the smaller circle is 15 cm and the radius of the larger circle is x cm larger.

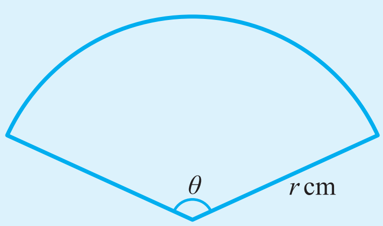
Find the value of x so that the area of the shaded region is 59.4 cm^2.



40 A piece of paper has a shape of a circular sector with radius r cm and angle θ radians. The paper is rolled into a cone with height 22 cm and base radius 8 cm.

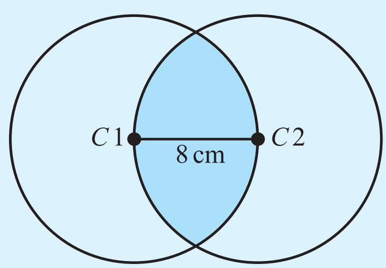
Find the values of r and θ.





41 Two identical circles each have radius 8 cm. They overlap in such a way that the centre of each circle lies on the circumference of the other, as shown in the diagram.

Find the perimeter and the area of the shaded region.



4B Further trigonometry

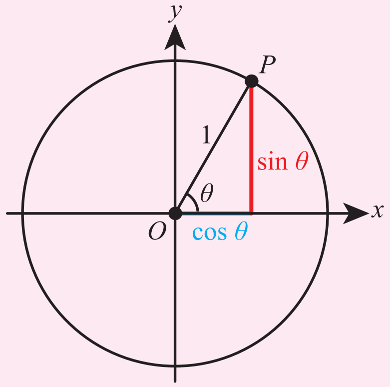
Definition of cos θ and sin θ in terms of the unit circle

Until now you have only used sin θ and cos θ in triangles, where θ had to be less than 180° (or π radians). However, using the unit circle we can define these functions so that θ can be any size, positive or negative.

KEY POINT 4.4

For a point, P, on the unit circle at an angle θ to the positive x-axis:

- sin θ is the y-coordinate of the point P
- cos θ is the x-coordinate of the point P.



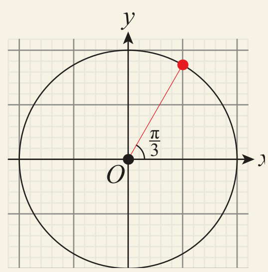
WORKED EXAMPLE 4.6

Mark on the unit circle the point corresponding to each angle θ. Hence find, or estimate, the values of sinθ and cosθ.

a $\theta = \frac{\pi}{3}$ **b** $\theta = \frac{7\pi}{2}$ **c** $\theta = -\frac{5\pi}{4}$

$\frac{\pi}{3} = \frac{1}{6}$ of 2π, so rotate one-sixth around the circle going anti-clockwise, starting from the positive x-axis. (In degrees, this corresponds to an angle of 60°.)

a



$\sin\left(\frac{\pi}{3}\right)$ is the y-coordinate and $\cos\left(\frac{\pi}{3}\right)$ is the x-coordinate

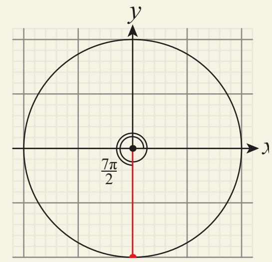
From this diagram, you can only estimate the values to 1 d.p.

$\sin\left(\frac{\pi}{3}\right) \approx 0.9$

$\cos\left(\frac{\pi}{3}\right) \approx 0.5$

b

$\frac{7\pi}{2} = 2\pi + \frac{3\pi}{2}$, so rotate one full turn plus another three-quarters of a turn anti-clockwise



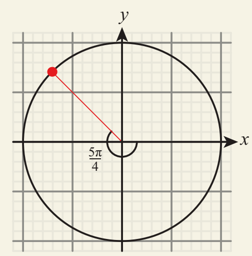
The point is on the y-axis, so its x-coordinate is zero

$\sin\left(\frac{7\pi}{2}\right) = -1$

$\cos\left(\frac{7\pi}{2}\right) = 0$

c

Negative sign means you need to rotate clockwise around the circle, in this case half a turn followed by another eighth. (Remember that an eighth of a turn corresponds to 45°)



You can see that the x-coordinate is negative but the y-coordinate is positive

$\sin\left(-\frac{5\pi}{4}\right) \approx 0.7$

$\cos\left(-\frac{5\pi}{4}\right) \approx -0.7$

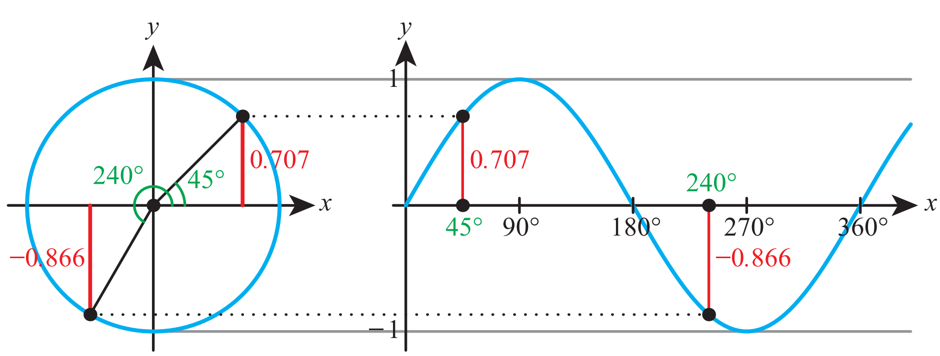
CONCEPTS – APPROXIMATION

For most angles, the values of sine and cosine found from the unit circle are only approximations. There are other methods for finding more accurate approximations. In a few special cases, you can use Pythagoras's Theorem to find the exact values.

For an architect, is it more useful to know that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ or 0.866?

Graphs of f(x) = sin(x) and f(x) = cos(x)

Using the definition of sine from the unit circle, we can draw its graph:

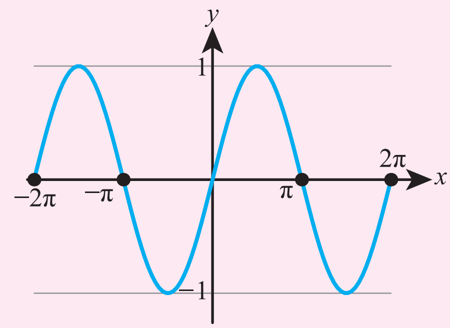


Sine repeats every 2π radians, so we say it is a periodic function with **period** 2π radians.

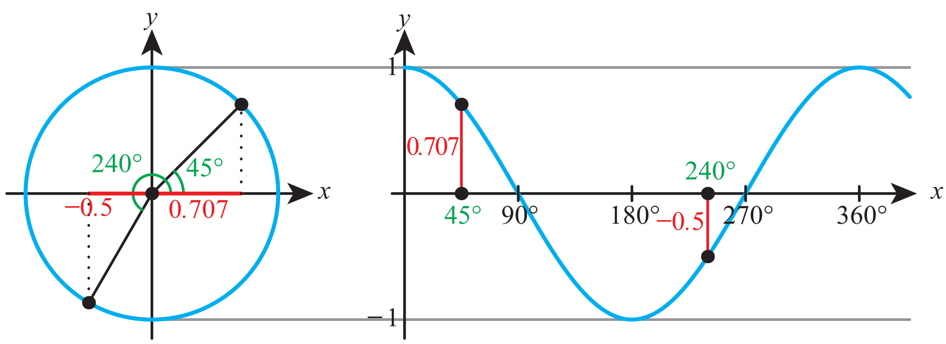
It has a minimum value of -1 and a maximum value of $+1$, so we say it has an **amplitude** of 1.

KEY POINT 4.5

The graph of $y = \sin x$:



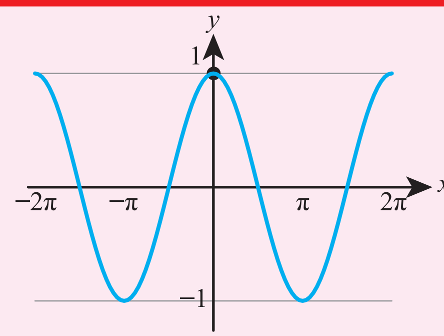
We can construct the cosine graph from the unit circle definition in the same way:



Like sine, cosine has a period of 2π radians and an amplitude of 1.

KEY POINT 4.6

The graph of $y = \cos x$:



Extension of the sine rule to the ambiguous case

You can see from the graph of sin(x) (or from the unit circle) that sin(θ) = (180° − θ) (or sin($\pi - \theta$) if working with radians). This means that, if you know the value of sinθ, there are two possible values between 0° and 180° that the angle θ could take.

This has an immediate implication for finding angles in triangles using the sine rule (where angles will usually be measured in degrees).

You used the sine rule to find angles (and side lengths) of triangles in Section 5B.

Tip

Note that there is not a corresponding relationship for cos, so you do not get this second possibility when using the cosine rule.

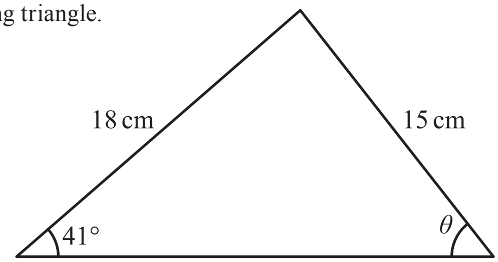
KEY POINT 4.7

When using the sine rule to find an angle, there may be two possible solutions: θ and $180 - \theta$.

Be aware that just because there is the possibility of a second value for an angle, this does not necessarily mean that a second triangle exists. You always need to check whether the angle sum is less than 180°.

WORKED EXAMPLE 4.7

Find the size of the angle θ in the following triangle.



Since we know a side and the angle opposite, the sine rule is useful

By sine rule, $\frac{\sin\theta}{18} = \frac{\sin 41}{15}$

This expression can be rearranged to find θ_1, the first possible value, which is the one given by the inverse sine function

$\theta_1 = \sin^{-1}\left(\frac{\sin 41}{15} \times 18\right)$

180 − 51.9 = 128.1° is also a possible value for θ

$\theta = 51.9°$ or $128.1°$

Check each possible value of θ to see that the angle sum is less than 180°

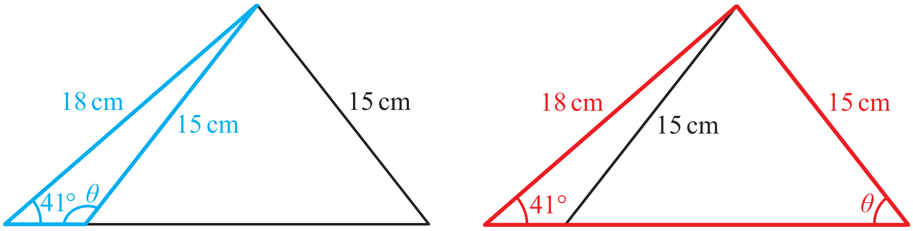
$51.9 + 41 = 92.9 < 180$

$128.1 + 41 = 169.1 < 180$

Both solutions are possible

So, $\theta = 51.9°$ or $128.1°$

The diagram below shows both possible triangles. Note that the 41° angle must remain opposite the 15 cm side, as given.



Be the Examiner 4.1

In triangle ABC, $AB = 10\text{ cm}$, $AC = 12\text{ cm}$ and $A\hat{B}C = 70°$.

Find the size of $A\hat{C}B$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\frac{\sin\theta}{10} = \frac{\sin 70°}{12}$ $\sin\theta = \frac{10\sin 70°}{12}$ $\theta = \sin^{-1}\left(\frac{10\sin 70°}{12}\right)$ $= 51.5°$ So, $\theta = 51.5°$	$\frac{\sin\theta}{12} = \frac{\sin 70°}{10}$ $\sin\theta = \frac{12\sin 70°}{10}$ $= 1.13$ $1.13 > 1$ So, there are no solutions.	$\frac{\sin\theta}{10} = \frac{\sin 70°}{12}$ $\sin\theta = \frac{10\sin 70°}{12}$ $\theta = \sin^{-1}\left(\frac{10\sin 70°}{12}\right)$ $= 51.5°$ $180 - 51.5 = 128.5$ So, $\theta = 51.5°$ or $128.5°$

Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$

The tangent function is defined as the ratio of the sine function to the cosine function.

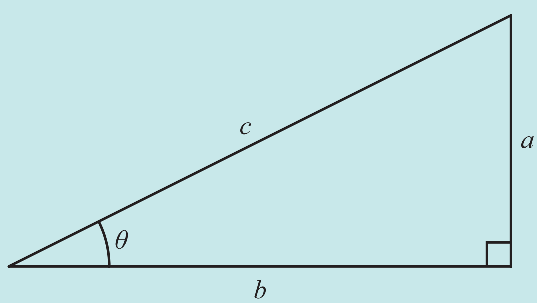
KEY POINT 4.8

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

For acute angles of θ this definition fits with the definition you already know based on right-angled triangles:

$\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$, so

$$\tan\theta = \frac{\left(\frac{a}{c}\right)}{\left(\frac{b}{c}\right)} = \frac{a}{b}$$

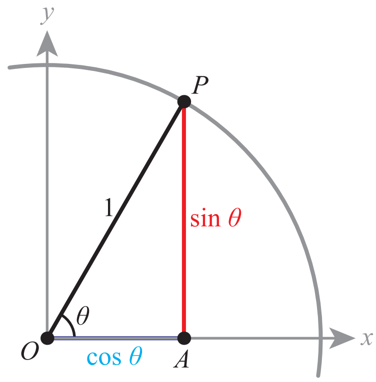


The Pythagorean identity

$$\cos^2\theta + \sin^2\theta \equiv 1$$

This identity, which relates cosine and sine, follows directly from their definitions on the unit circle:

Using Pythagoras in the right-angled triangle *OAP* gives the identity.



KEY POINT 4.9

$\cos^2\theta + \sin^2\theta \equiv 1$

WORKED EXAMPLE 4.8

Given that $\frac{3\pi}{2} < x < 2\pi$ and that $\cos x = \frac{1}{3}$, find the exact value of

a $\sin x$ **b** $\tan x$

Use $\cos^2 x + \sin^2 x \equiv 1$ to relate sin to cos

a $\sin^2 x \equiv 1 - \cos^2 x$

$= 1 - \left(\frac{1}{3}\right)^2$

Substitute in the given value of $\cos x$

$= 1 - \frac{1}{9}$

$= \frac{8}{9}$

Remember to take ± when square rooting

$\sin x = \pm\sqrt{\frac{8}{9}}$

Looking at the unit circle or the graph, you can see that $\sin x$ is negative for x between $\frac{3\pi}{2}$ and 2π

But $\sin x < 0$ for $\frac{3\pi}{2} < x < 2\pi$, so

$\sin x = -\sqrt{\frac{8}{9}}$

You may be able to use the calculator to simplify the surd

$= -\frac{2\sqrt{2}}{3}$

Use $\tan x = \frac{\sin x}{\cos x}$ to relate tan to sin and cos

b $\tan x = \frac{\sin x}{\cos x}$

Substitute in the values for $\sin x$ and $\cos x$ and simplify

$= \frac{\left(-\frac{2\sqrt{2}}{3}\right)}{\left(\frac{1}{3}\right)}$

$= -2\sqrt{2}$

Solving trigonometric equations in a finite interval graphically

You can solve a trigonometric equation graphically in the same way that you have solved other equations with your GDC. Because trigonometric functions are periodic, you will often find more than one solution.

See Chapter 3 of the Mathematics: applications and interpretation SL book for a reminder of how to use your GDC to solve equations.

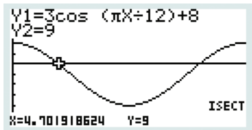
Tip

When using trigonometric models, you should work in radians unless told otherwise.

WORKED EXAMPLE 4.9

The depth of water (d m) in harbour, t hours after midnight, can be modelled by the equation $d = 3\cos\left(\frac{\pi}{12}t\right) + 8$. Find the times at which the depth of water in the harbour is 9 m, giving your answer to the nearest minute.

Use your GDC to find the intersection of $y = 3\cos\left(\frac{\pi}{12}x\right) + 8$ and $y = 9$



$t = 4.702, 19.298$

Convert 0.702 and 0.298 to minutes by multiplying by 60

So at 04 : 42 and 19 : 18

You studied sinusoidal models in Chapter 13 of the Mathematics: applications and interpretation SL book and will revisit them in Chapter 6.

Exercise 4B

In questions 1 to 4 use the method demonstrated in Worked Example 4.6 to mark the point corresponding to angle θ on the unit circle, and hence estimate the values of $\sin\theta$ and $\cos\theta$.

1 **a** $\theta = \frac{2\pi}{3}$ **b** $\theta = \frac{\pi}{4}$

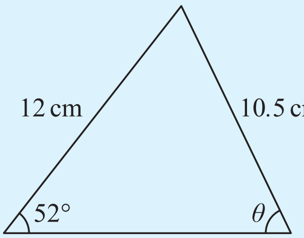
2 **a** $\theta = \pi$ **b** $\theta = 4\pi$

3 **a** $\theta = \frac{9\pi}{2}$ **b** $\theta = \frac{11\pi}{2}$

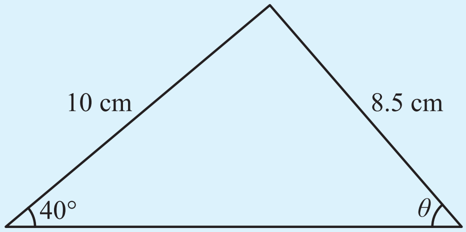
4 **a** $\theta = -\frac{5\pi}{6}$ **b** $\theta = -\frac{9\pi}{4}$

In questions 5 to 8 use the method demonstrated in Worked Example 4.7 to find the possible size(s) of the angle θ in each triangle. Give your answers correct to the nearest degree.

5 **a**



b



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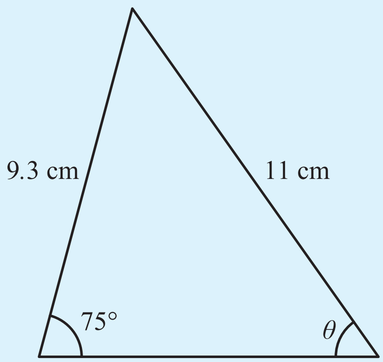

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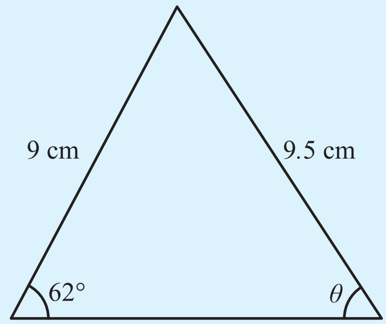

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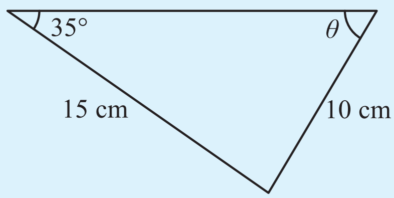

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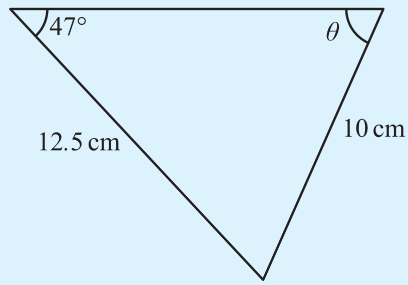

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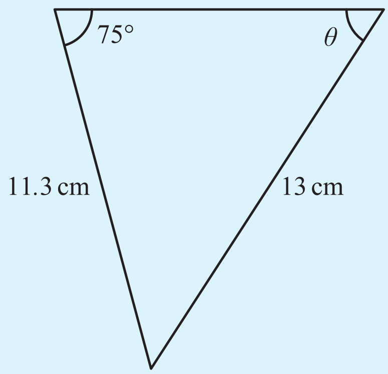

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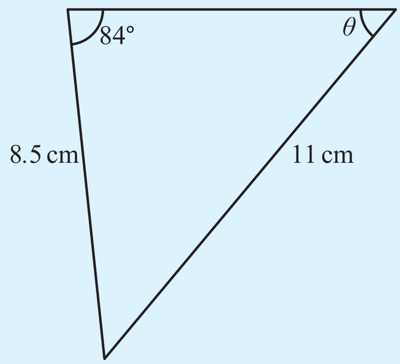


In questions 9 to 12 use the method demonstrated in Worked Example 4.8 to find the possible values of the required trigonometric functions.

9 **a** Find $\sin x$ and $\tan x$ given that $\cos x = \frac{1}{4}$ and $0 < x < \frac{\pi}{2}$

b Find $\sin x$ and $\tan x$ given that $\cos x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$

10 **a** Find $\cos x$ and $\tan x$ given that $\sin x = 0.381$ and $\frac{\pi}{2} < x < \pi$

b Find $\cos x$ and $\tan x$ given that $\sin x = 0.722$ and $0 < x < \frac{\pi}{2}$

11 **a** Find $\sin x$ and $\tan x$ given that $\cos x = -0.062$ and $\pi < x < \frac{3\pi}{2}$

b Find $\sin x$ and $\tan x$ given that $\cos x = -0.831$ and $\frac{\pi}{2} < x < \pi$

12 **a** Find $\cos x$ and $\tan x$ given that $\sin x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$

b Find $\cos x$ and $\tan x$ given that $\sin x = -\frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$

In questions 13 to 16 use a graphical method, as illustrated in Worked Example 4.9, to find all solutions of the equation in the given interval.

13 a $3\sin x = 0.5$ for $0 \le x < 3\pi$

b $2\cos x = -0.2$ for $0 \le x \le 4\pi$

14 a $2 + 5\cos(3x) = 4$ for $0 \le x \le \pi$

b $3 + 4\cos(2x) = 2$ for $0 \le x \le \pi$

15 a $4 - 2\sin(2x) = 1$ for $0 \le x \le 360°$

b $3\sin(4x) - 5 = -1$ for $0 \le x \le 180°$

16 a $4\tan(\pi(x-2)) = 3$ for $-1 < x < 1$

b $3\tan\left(\frac{x-\pi}{2}\right) = 5$ for $0 < x < 4\pi$

17 Given that $\sin\theta = -\frac{4}{9}$, find the possible values of $\cos\theta$.

18 Given that $\cos\theta = \frac{2}{5}$ and $\pi < \theta < \frac{2\pi}{3}$, find the exact value of

a $\sin\theta$

b $\tan\theta$

19 Given that $\sin x = \frac{3}{7}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of

a $\cos x$

b $\tan x$

20 The depth of water in a harbour varies according to the equation $d = 5 + 1.6\sin\left(\frac{\pi}{12}t\right)$, where d is measured in metres and t is the time, in hours, after midnight.

a Find the depth of water at high tide.

b Find the first time after midnight when the high tide occurs.

21 The height of a seat on Ferris wheel above ground is modelled by the equation $h = 6.2 - 4.8\cos\left(\frac{\pi}{4}t\right)$, where h is measured in metres and t is the time, in minutes, since the start of the ride.

a How long does the wheel take to complete one revolution?

b Find the maximum height of the seat above ground.

c Find the height of the seat above ground 2 minutes and 40 seconds after the start of the ride.

22 A ball is attached to one end of a spring and hangs vertically. The ball is then pulled down and released. In subsequent motion, the height of the ball, h metres, above ground at time t seconds is given by $1.4 - 0.2\cos(15t)$.

a Find the greatest height of the ball above ground.

b How many full oscillations does the ball perform during the first 3 seconds?

c Find the second time when the ball is 1.5 m above ground.

23 In triangle ABC, $AB = 8\text{cm}$, $AC = 11\text{cm}$ and angle $BAC = \theta°$. The area of the triangle is 35cm^2. Find the possible values of θ.

24 In triangle ABC, $AB = 7.5\text{cm}$, $BC = 5.3\text{cm}$ and angle $BAC = 44°$. Find the two possible values of the length AC.

25 Triangle KLM has $KL = 12\text{cm}$, $LM = 15\text{cm}$ and angle $MLK = 55°$. Show that there is only one possible value for the length of the side KM and find this value.

26 Express $3\sin^2 x + 7\cos^2 x$ in terms of $\sin x$ only.

27 Express $4\cos^2 x - 5\sin^2 x$ in terms of $\cos x$ only.

28 Prove the identity $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 \equiv 2$.

29 Prove that $1 + \tan^2\theta \equiv \frac{1}{\cos^2\theta}$.

30 Prove that $\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} \equiv 1$.

31 Prove the identity $\frac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$.

32 It is given that $2\cos^2 x + \sin x = k$, where $1 \le k \le 2$. Find, in terms of k, the possible values of $\sin x$.

4C Matrices as transformations

Tip

Notice that the coordinates of a point are written as a position vector and go after the matrix.

You learnt more about position vectors in Chapter 2.

You already know how to draw the image of an object after a transformation such as rotation, reflection or enlargement. It can be more difficult to calculate its exact coordinates. For example, what are the coordinates of the image when the point (4, 1) is reflected in the line $y = 3x$? In this section you will learn how to use matrices to do this.

If a transformation is represented by a matrix **M**, then the image of the point with coordinates (x, y) is given by $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$. For example, if a transformation has matrix $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ then the point with coordinates (x, y) is mapped to the point with coordinates $(3x + y, x + 2y)$, because

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x + 2y \end{pmatrix}$$

WORKED EXAMPLE 4.10

A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$.

a Find the coordinates of the image of the point (3, −2).

b Find the coordinates of the point whose image is (1, 3).

The image of the point (x, y) is given by $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$

a $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$

The coordinates of the image point are (12, 1).

You can use inverse matrix to find the point with the given image

b $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

You can find inverse matrix using technology

$\mathbf{M}^{-1} = \begin{pmatrix} 0.4 & 0.6 \\ -0.2 & 0.2 \end{pmatrix}$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ -0.2 & 0.4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The coordinates of the original point are (2, 1).

How can we find matrices to represent common transformations, such as rotations or reflections? Conversely, if you are given a matrix, can you describe geometrically the transformation it represents?

A key observation is that the images of the points (1, 0) and (1, 0) correspond to the columns of the matrix. For example, for the transformation with the matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, we have:

$$\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{M}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Tip

The origin (0, 0) is always mapped to itself under a matrix transformation.

KEY POINT 4.10

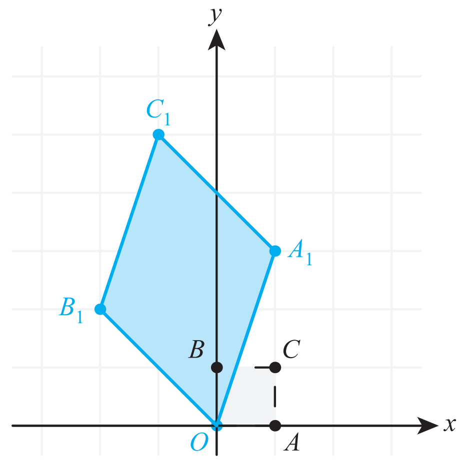
For a transformation represented by a matrix **M**, the image of the point (1, 0) is the first column of **M** and the image of the point (0, 1) is the second column of **M**.

To visualise the effect of the transformation it is often useful to look at the image of the **unit square**, the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).

WORKED EXAMPLE 4.11

a Draw the unit square and its image under the transformation with matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

b The diagram shows the unit square and its image under a transformation. Find the matrix representing the transformation.

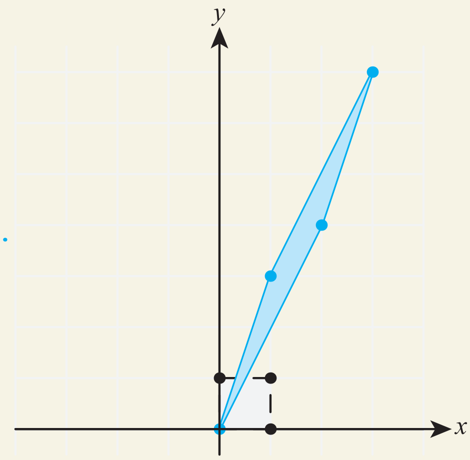


Find the coordinates of the images of the vertices (0, 0), (1, 0), (1, 1) and (0, 1). (You can use the fact that the images of (1, 0) and (0, 1) are the columns of the matrix

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

The unit square is drawn with dashed lines and its image with solid lines



The first column is the image of the point $A(1, 0)$, which is $A_1(1, 3)$

b $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$

The second column is the image of the point $B(0, 1)$, which is $B_1(-2, 2)$

Some common transformations

You can use the result of Key Point 4.10 to find matrices representing common transformations.

KEY POINT 4.11

Matrices representing common transformations:

reflection in the line $y = (\tan\theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
horizontal stretch with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch with scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
enlargement with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
anticlockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
clockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

There are several things you should note about using these matrices:

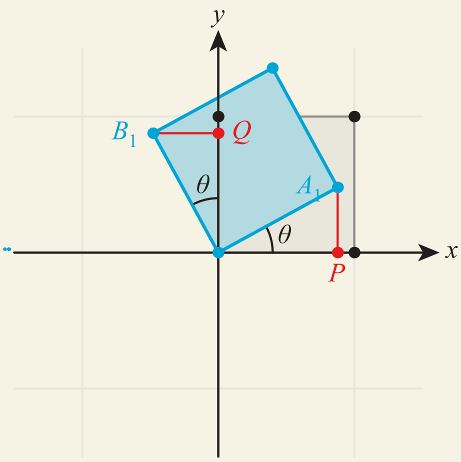
- The reflection line $y = (\tan\theta)x$ passes through the origin and makes angle θ with the positive x-axis. You will often need to use special cases such as the lines $y = x$ (where $\theta = \frac{\pi}{4}$) and $y = -x$ (where $\theta = \frac{3\pi}{4}$).
- Make sure you distinguish between a stretch (which changes only one of x or y coordinates) and an enlargement (which changes both).
- In the matrices for the rotation, the angle θ needs to be positive.

The transformation matrices in Key Point 4.11 and an explanation of their respective actions are all given in the formula book.

Proof 4.1

Prove that the matrix representing an anti-clockwise rotation of angle θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Draw a diagram showing the images of the points (1, 0) and (0, 1)



Triangle OPA_1 is right-angled with hypotenuse $OA_1 = 1$(as this is the same as the length of OA). You can use trigonometry to find the lengths OP and PA_1

$OP = \cos\theta,\ PA_1 = \sin\theta$

so

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Triangle OQB_1 is also right-angled with hypotenuse $OB_1 = 1$, so you can find the lengths OQ and OB_1. You can see from the diagram that point B_1 has a negative x-coordinate and a positive y-coordinate

$OQ = \sin\theta,\ QB_1 = \cos\theta$

so

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

The images of (1,0) and (0,1) give the coordinates of the transformation matrix

Hence the matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

WORKED EXAMPLE 4.12

Write down the matrix representing each transformation, and hence find the image of the point (–2, 3).

a Vertical stretch with scale factor 4.8.

b Clockwise rotation of angle 60° about the origin.

c Reflection in the line $y = 2x$.

You can use the table in Key Point 4.11, but you can also remember that a vertical translation only affects y-coordinates, so the point (0,1) is mapped to (0,4.8)

a $\begin{pmatrix} 1 & 0 \\ 0 & 4.8 \end{pmatrix}$

$\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$. Note that the rotation is clockwise

b $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

The gradient of the line is $\tan\theta$

c $\tan\theta = 2 \Rightarrow \theta =$

$\mathbf{M} =$

You need to find $\cos 2\theta$ and $\sin 2\theta$:

$\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

The table in Key Point 4.11 contains all the familiar transformations except for translations. A translation cannot be achieved by matrix multiplication, because it moves the origin. Instead, a translation is determined by its translation vector. To translate a point, you simply add the translation vector to the point's position vector.

[Author query: Should T and S be italic?]

WORKED EXAMPLE 4.13

Transformation T is a reflection in the x-axis and transformation S is a translation with vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Find the image of the point (2, –3) after the following sequence of transformations:

a T followed by S

b S followed by T.

The reflection in the x-axis transforms (1,0) to (1,0) and (0,1) to (0,–1), so the matrix for T is $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Apply the reflection matrix to the position vector of (2, –3) and then add the translation vector

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

The image of (2, –3) is (1, 6).

This time add the translation vector first, then apply the reflection matrix

b $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\left(\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}\right)$

$=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

The image of (2, −3) is (1, −1).

Composition of transformations

As you can see from Worked Example 4.13, the result of a composition of two transformations depends on the order in which they are applied.For two transformation represented by matrices, the combined transformation is represented by the product of the two matrices.

KEY POINT 4.12

If the transformation with matrix **A** is followed by the transformation with matrix **B**, the combined transformation has matrix **BA**.

Notice the order of the matrices in the product: if **A** is applied to a vector x first the image is the vector **Ax**. When transformation **B** is applied to this image, the final image is $\mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{BA})\mathbf{x}$.

WORKED EXAMPLE 4.14

Find the matrix representing a 45° anti-clockwise rotation about the origin followed by a reflection in the line $y = x$.

Find the matrix for the first transformation. It is an anti-clockwise rotation so use $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$\mathbf{A}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$

The second transformation is a reflection in the line $y = x$, so use $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$ where $\theta = 45°$

$\mathbf{B}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

The combined transformation (**A** followed by **B**) has matrix **BA**

$\mathbf{BA}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$

You are the Researcher

Two examples of fractals are given in questions 49 and 50 of Exercise 4C. Investigate other examples of fractals, such as the Koch Snowflake and the Sierpinski Triangle (shown).

04_61

An important example of a composition of transformations is repeatedly combining a transformation with itself. This is written as $\mathbf{A}^2$, $\mathbf{A}^3$, etc. Applying a transformation repeatedly can generate interesting shapes known as **fractals**.

Be the Examiner 4.2

Transformations **A** and **B** have matrices $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$. Transformation **C** is **B** followed by **A**. Find the matrix of the transformation **C**.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$	$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$	$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$
$= \begin{pmatrix} 4 & 1 \\ -1 & 7 \end{pmatrix}$	$= \begin{pmatrix} 5 & 1 \\ -3 & 12 \end{pmatrix}$	$= \begin{pmatrix} 5 & 3 \\ -1 & 12 \end{pmatrix}$

The determinant of a transformation matrix

When a shape is transformed using a matrix transformation, there is a simple way to find the area of the image.

You learnt how to find the determinant of a 2×2 matrix in Section 4B.

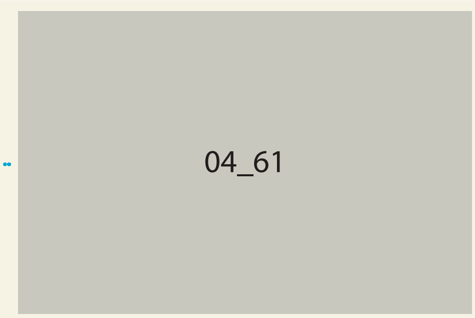
KEY POINT 4.13

For a transformation represented by matrix A, area of image $= |\det \mathbf{A}| \times$ area of object

WORKED EXAMPLE 4.15

The triangle with vertices (1, 2), (5, 2) and (2, 5) is transformed using the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$. Find the area of the image triangle.

The original triangle has a horizontal base, so it is easy to find its area



$$\text{Area} = \frac{1}{2}(4)(3) = 6$$

Find the determinant of the matrix

$$\det \mathbf{A} = -2 - 5 = -7$$

Remember to take the modulus of the determinant

So

$$\text{image area} = 7 \times 6 = 42$$

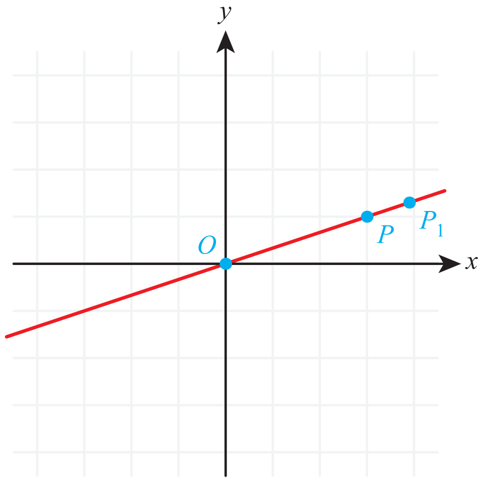
Tip

Transformations with a negative determinant reverse the orientation of the object.

Looking at the table in Key Point 4.11, you can see that rotation matrices always have determinant 1 and reflection matrices always have determinant –1. This corresponds to the fact that they do not change the size of the object.

Eigenvalues and eigenvectors

You know from Section 4D that, for a matrix **A**, an eigenvector **v** with eigenvalue λ satisfies $\mathbf{Av} = \lambda\mathbf{v}$. Geometrically this means that, if point P has position vector $\overrightarrow{OP} = \mathbf{v}$ then its image P_1 (under the transformation with matrix **A**) has position vector $\overrightarrow{OP_1} = \lambda\mathbf{v} = \lambda\overrightarrow{OP}$. Hence any point on the line through the origin in the direction of vector **v** is mapped to another point on the same line (with the distance from the origin increasing by a factor λ).



We say that the line OP is *invariant* under the transformation. If the eigenvalue $\lambda = 1$, each point on the line is actually mapped to itself.

WORKED EXAMPLE 4.16

Using eigenvectors,

a find the invariant lines for the horizontal stretch with scale factor 3

b show that a rotation through 90° anti-clockwise about the origin has no invariant lines.

a
$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

First find the eigenvalues of the transformation matrix

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$\det\begin{pmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0$$
$$(3-\lambda)(1-\lambda) = 0$$
$$\lambda = 1, 3$$

Now find the eigenvector associated with each eigenvalue

When $\lambda = 1$,
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{cases} 3x = x \Rightarrow x = 0 \\ y = y \end{cases}$$

You can choose any value for y; choose $y = 1$

The eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

When $\lambda = 3$,

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x = 3x \\ y = 3y \Rightarrow y = 0 \end{cases}$$

The eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The invariant lines are the lines through the origin in the direction of each eigenvector

The invariant lines are $y = 0$ and $x = 0$.

b $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Again, start by trying to find the eigenvalues of the transformation matrix

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

The characteristic equation has no real roots

No real roots, so the transformation matrix has no real eigenvalues and hence no invariant lines.

You are the Researcher

In Worked Example 4.16, you found that the horizontal stretch with scale factor 3 has two eigenvectors: one in the direction of the x-axis with eigenvalue 3, and one in the direction of the y-axis with eigenvalue 1. Geometrically, this corresponds to the fact that the points on the x-axis are moved three times further away from the origin, while the points on the y-axis stay fixed. Use eigenvalues and eigenvectors to investigate invariant lines of other common transformations and their combinations.

Exercise 4C

In questions 1 to 3 use the method demonstrated in Worked Example 4.10a to find the coordinates of the image of the given point under the given transformation.

1 **a** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, point (2,3)

b $\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, point (4,2)

2 **a** $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$, point (2,1)

b $\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix}$, point (1,3)

3 **a** $\begin{pmatrix} 4 & 1 \\ -1 & -5 \end{pmatrix}$, point (2, −3)

b $\begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix}$, point (−2, 5)

In questions 4 to 6 use the method demonstrated in Worked Example 4.10b to find the coordinates of the point whose image under the given transformation is given.

4 **a** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, image point (6,8)

b $\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, image point (13,5)

5 **a** $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$, image point (–6, –7)

b $\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix}$, image point (–8, –3)

6 **a** $\begin{pmatrix} 4 & 1 \\ -1 & -5 \end{pmatrix}$, image point (–2, 10)

b $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, image point (6,3)

In questions 7 to 10 use the method demonstrated in Worked Example 4.11a to draw the image of the unit square under the given transformation.

7 **a** $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

b $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

8 **a** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

b $\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$

9 **a** $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

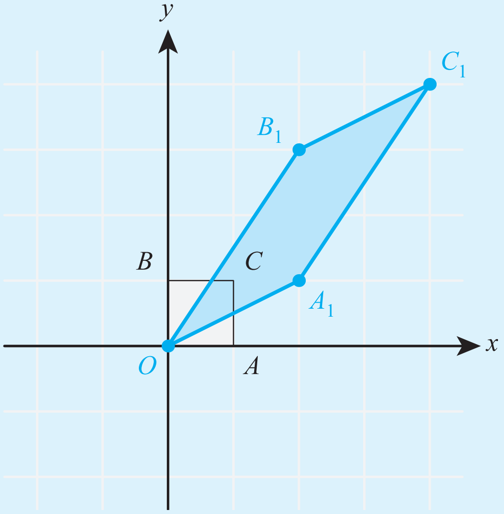
b $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

10 **a** $\begin{pmatrix} 1 & -2 \\ -4 & 1 \end{pmatrix}$

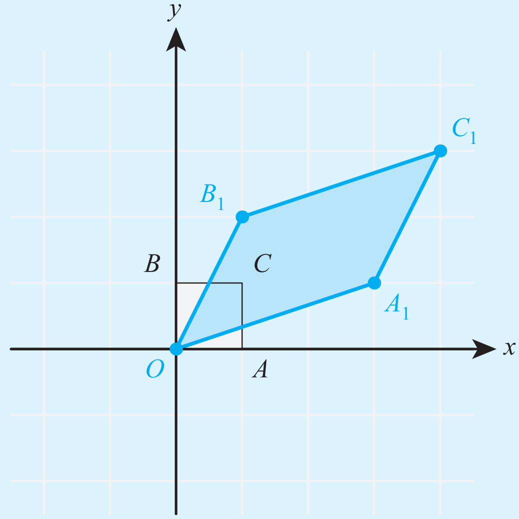
b $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

In questions 11 to 16 each diagram shows the unit square and its image under a transformation. Use the method demonstrated in Worked Example 4.11b to find the matrix representing the transformation.

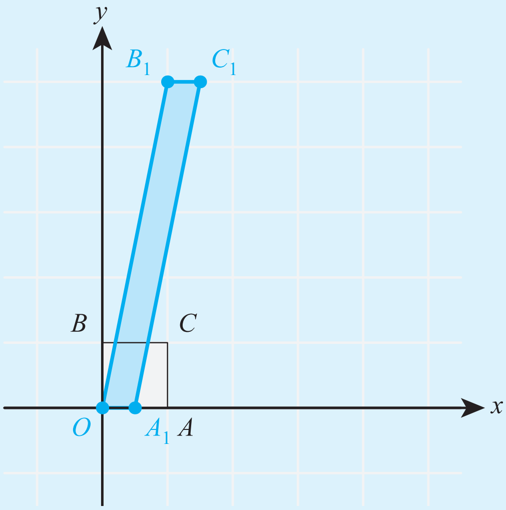
11 **a**



b



12 a



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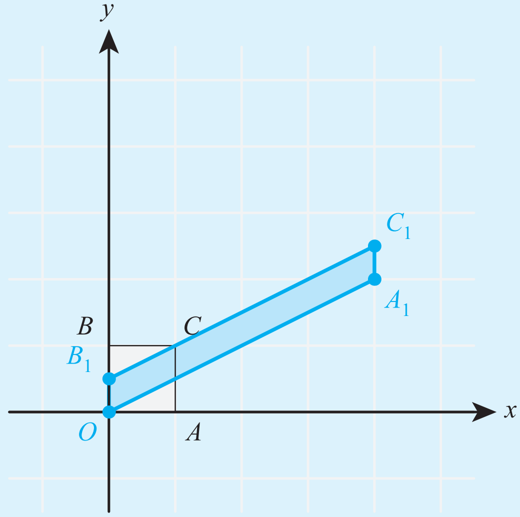
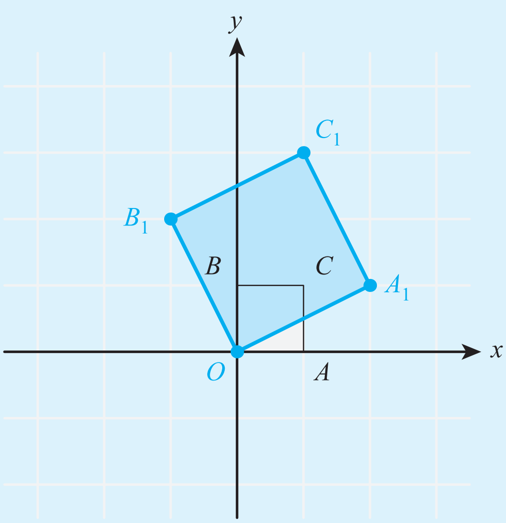
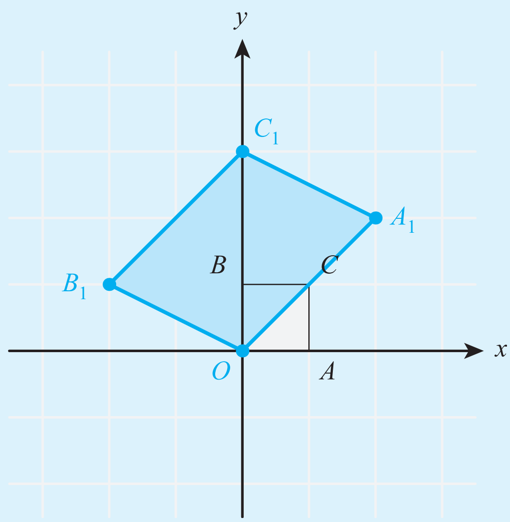


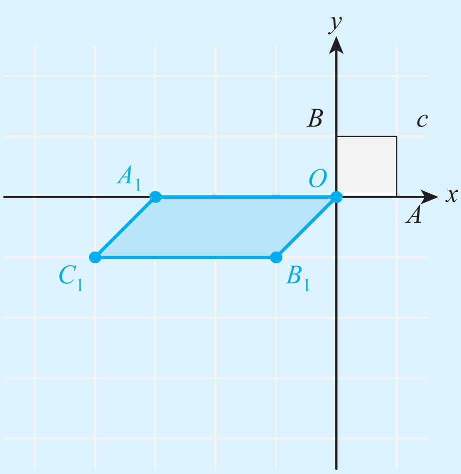
13 a



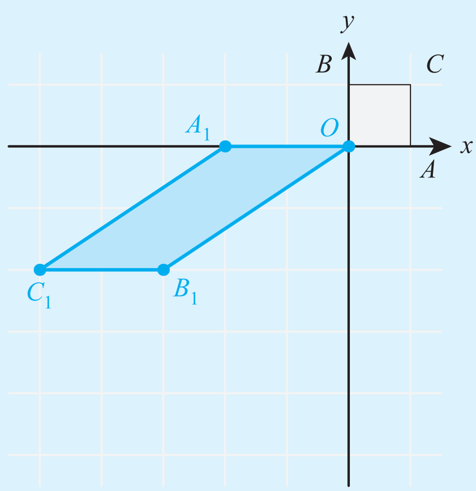
b



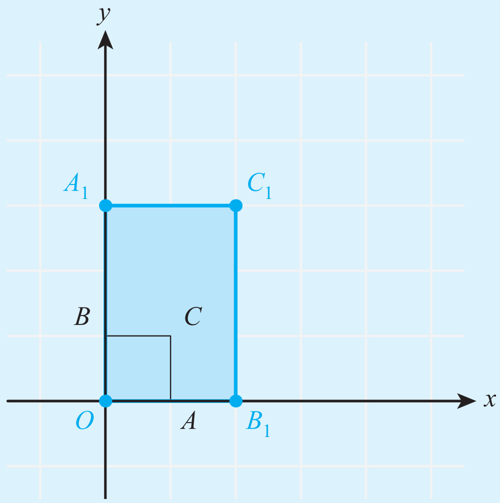
14 a



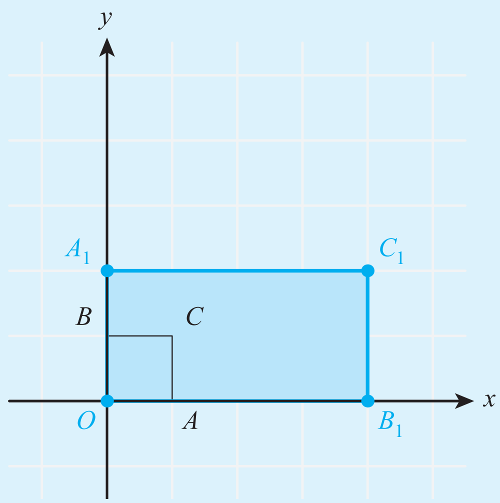
b



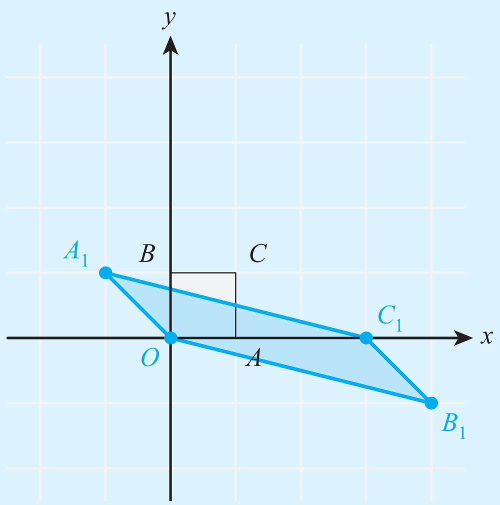
15 **a**



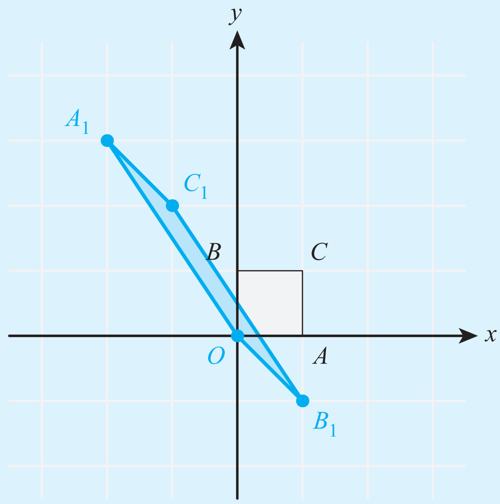
b



16 **a**



b



In questions 17 to 24 use the method demonstrated in Worked Example 4.12 to write down the matrix representing each transformation.

17 **a** Horizontal stretch with scale factor 3

b Vertical stretch with scale factor 2

18 **a** Vertical stretch with scale factor $\frac{1}{4}$

b Horizontal stretch with scale factor $\frac{1}{3}$

19 **a** Enlargement with scale factor 5

b Enlargement with scale factor 6

20 **a** Reflection in the line $y = x$

b Reflection in the line $y = -x$

21 **a** Reflection in the line $y = \sqrt{3}x$

b Reflection in the line $y = \frac{1}{\sqrt{3}}x$

22 **a** Rotation 30° anti-clockwise about the origin

b Rotation 45° anti-clockwise about the origin

23 **a** Rotation 90° clockwise about the origin

b Rotation 180° about the origin

24 **a** Rotation 120°clockwise about the origin

b Rotation 150° clockwise about the origin

In questions 25 to 27, use the method demonstrated in Worked Example 4.13 to find the image of the given point under the given sequence of transformations.

25 **a** Point $(2, -3)$; rotation 90° anti-clockwise about the origin followed by translation with vector $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

b Point $(2, 4)$; reflection in the y-axis followed by translation with vector $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

26 **a** Point $(1, 1)$; translation with vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ followed by an enlargement with scale factor 3

b Point $(0, 6)$; translation with vector $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ followed by a horizontal stretch with scale factor 4

27 **a** Point $(1, -2)$; translation with vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ followed by reflection in the line $y = x$ followed by translation with vector $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

b Point $(3, -1)$; translation with vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ followed by rotation 90° clockwise about the origin followed by translation with vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

In questions 28 to 30, use the method demonstrated in Worked Example 4.14 to find the matrix representing the composite transformation.

28 **a** Transformation with matrix $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ followed by the transformation with matrix $\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ followed by the transformation with matrix $\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$

29 **a** Rotation 90° anti-clockwise about the origin followed by the transformation with matrix $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}$ followed by the reflection in the line $y = x$

30 **a** Reflection in the x-axis followed by the reflection in the line $y = -x$

b Rotation 180° followed by reflection in the y-axis

In questions 31 to 34, the triangle with vertices $(1, 0)$, $(4, 0)$ and $(2, 4)$ is transformed by the given transformation. Use the method demonstrated in Worked Example 4.15 to find the area of the image.

31 **a** Vertical stretch with scale factor 3

b Horizontal stretch with scale factor 0.2

32 **a** Enlargement with scale factor $\frac{1}{2}$

b Enlargement with scale factor -2

33 **a** Rotation 90° clockwise about the origin

b Reflection in the x-axis

34 **a** Transformation with matrix $\begin{pmatrix} 3 & -2 \\ -4 & 2 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

35 A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$.

a Find the coordinates of the image of the point $(-1, 2)$ under this transformation.

b Draw the image of the unit square.

c Find the area of the image of the unit square.

36 The transformation with matrix $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$ maps the point with coordinates $(3, p)$ to the point with coordinates $(q, -1)$.

a Find the values of p and q.

b Find the area of the image of the triangle with vertices $(0, 0)$, $(6, 0)$ and $(2, 5)$.

c Find the image of the point $(2, 3)$ under the transformation with matrix $\mathbf{A}^3$.

37 Transformation **R** is a rotation 90° clockwise around the origin. Transformation **T** is a translation with vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

a Transformation **R** maps point (p, q) to the point $(3, -2)$. Find the values of p and q.

b Point (a, b) is mapped using transformation **R** followed by transformation **T**. Find the coordinates of the image in terms of a and b.

c Point (a, b) is now mapped using transformation **T** followed by transformation **R**. Show that the final image is different from the one in part **b**.

38 a Find the image of the point $(2, -3)$ under a rotation through 135° anti-clockwise about the origin.

b Find the image of the point $(2, -3)$ under a rotation through 135° anti-clockwise about the origin, followed by an enlargement with scale factor 4.

c Find the area of the unit square after the sequence of transformations from part **b**.

39 a Find a 2×2 matrix representing reflection in the line $y = 3x$.

b Find the coordinates of the point whose image under the reflection is $(2, 2)$.

40 Transformation **P** has matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and transformation **Q** has matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

a Find the image of the point $(3, 1)$ under the transformation **P** followed by the transformation **Q**.

b Point C is transformed using transformation **P** followed by the transformation **Q**. The image has coordinates $(-2, 2)$. Find the coordinates of C.

41 Let **N** be an enlargement with scale factor 2 centered at the origin, and let **R** be the rotation through 30° about the origin.

a The point $(2, 1)$ is transformed using the rotation followed by the enlargement. Find the coordinates of the image.

b Find the matrix representing the transformation resulting from the enlargement followed by the rotation.

c Show that **N** followed by **R** always gives the same result as **R** followed by **N**.

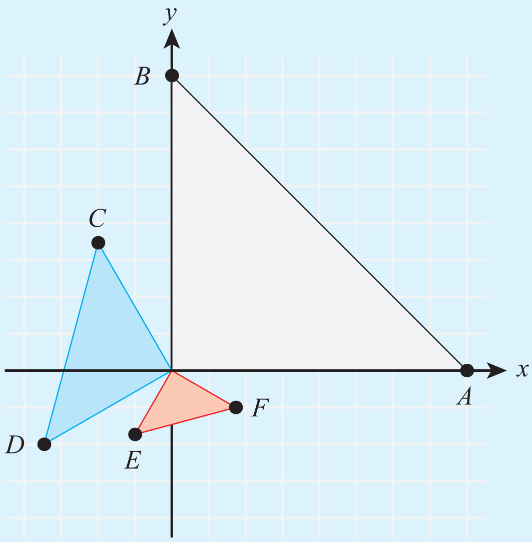
42 Use matrices to prove that rotation through 30° about the origin followed by rotation through 60° about the origin (in the same direction) results in the rotation through 90° about the origin.

43 Let **S** be the reflection in the x-axis and **R** be the rotation 90° anti-clockwise about the origin.

a Show that **S** followed by **R** results in a reflection and find the equation of its mirror lines.

b Show that **R** followed by **S** results in a different reflection.

44 A logo is designed by rotating and shrinking triangle OAB, as shown in the diagram, where the coordinates of the vertices are $A(8, 0)$ and $B(0, 8)$.



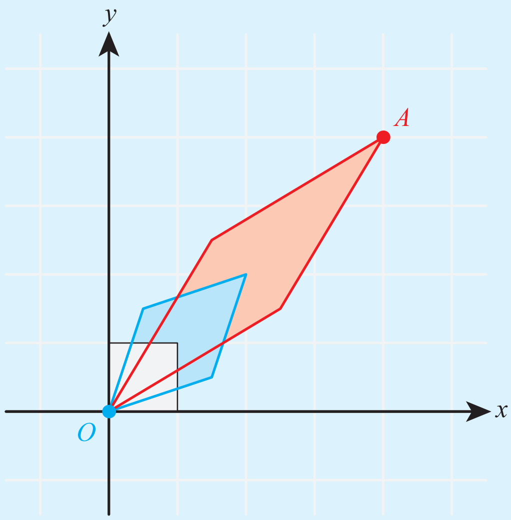
Let **R** denote rotation 120° anti-clockwise around the origin, and **S** an enlargement with scale factor $\frac{1}{2}$. Let **T** be the composite transformation **R** followed by **S**.

a Find the matrices representing **T** and $\mathbf{T}^2$.

b Find the coordinates of D and F.

c Find the area of the whole logo.

45 Transformation **M** has matrix $\mathbf{M} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$. The diagram shows the unit square and its images under **M** and $\mathbf{M}^2$.



a Find the coordinates of the point **A**.

b Find the area of the parallelogram which is the image of the unit square under $\mathbf{M}^6$.

46 Let **S** be the matrix representing a reflection in the line $y = -x$.

a Write down the matrix **S**.

b Find the eigenvectors and corresponding eigenvalues of **S**.

c Interpret geometrically the meaning of the two eigenvectors.

47 a Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{B}_1 = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$.

b Hence determine the equations of the invariant lines of the form $y = mx$ of the transformation represented by $\mathbf{B}_1$.

c Show that the transformation represented by the matrix $\mathbf{B}_2 = \begin{pmatrix} -5 & -3 \\ 4 & 1 \end{pmatrix}$ has no invariant lines passing through the origin.

48 **R** is a reflection in the line $y = 2x$.

S is a 90° rotation anticlockwise about the origin.

T the transformation '**R** followed by **S**'.

a Find the matrix representing **T**.

b Hence find the equations of invariant lines of **T** which pass through the origin.

c Show that **T** is a reflection and find the equation of the mirror line.

49 Transformation **T** is an enlargement with scale factor $\frac{1}{2}$ followed by a translation with vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

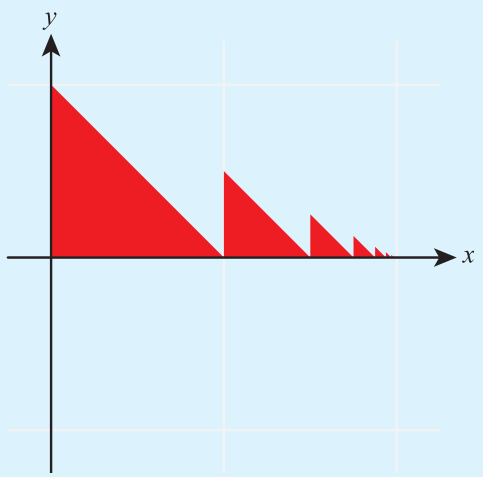
a Write down the coordinates of the image of the point $P_0(x, y)$.

Let P_n be the image of P_0 under the transformation $\mathbf{T}^n$.

b Find the coordinates of P_2, P_3 and P_4.

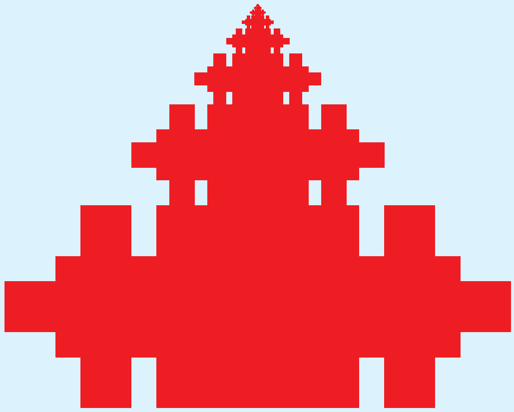
c Hence conjecture an expression for the coordinates of P_n.

The diagram shows a design formed by repeatedly applying transformation **T** to the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.



d Find the total area of the design.

50



You will need the formula for the sum of an infinite geometric series, which you met in Section 1C.

The fractal shown in the diagram is formed as follows:

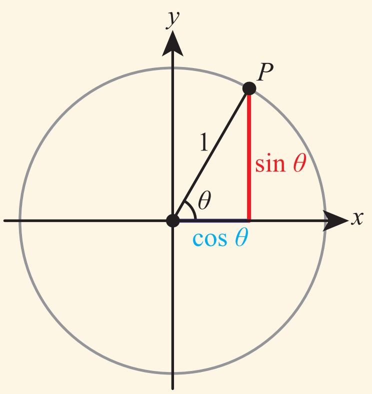
1 Start with a square of side 1.
2 At step 1, construct three squares of side $\frac{1}{2}$ on three of the sides of the original square.
3 At each subsequent step, construct nine new squares, on the middle half of the sides of three of the squares from the previous step.

a Write down the length of the side of each square constructed at step **4**.
b Find the total area of the fractal.

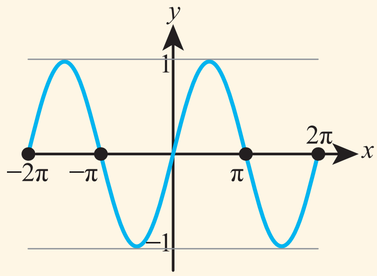
Checklist

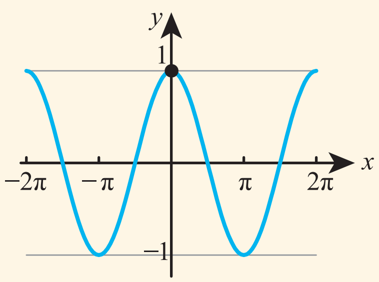
- You should be able to convert between degrees and radians $360° = 2\pi$ radians
- You should be able to find the length of an arc of a circle $s = r\theta$ where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.
- You should be able to find the area of a sector of a circle $A = \frac{1}{2}r^2\theta$ where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.
- You should be able to define the sine and cosine functions in terms of the unit circle.

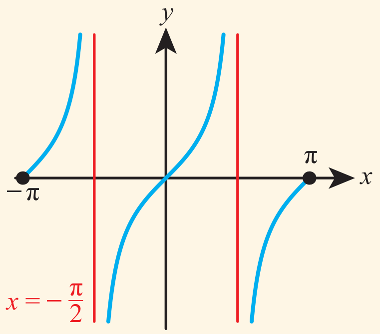
  For a point P on the unit circle,
  - $\sin\theta$ is the y-coordinate of the point P
  - $\cos\theta$ is the x-coordinate of the point P



- You should be able to define the tangent function $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- You should be able to sketch the graphs of trigonometric functions:







- You should know about the ambiguous case of the sine rule:

  When using the sine rule to find an angle, there may be two possible solutions: θ and $180-\theta$.
- You should know the Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$
- You should be able to solve trigonometric equations graphically.
- You should be able to use matrices to represent transformations of the form $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.
- You should be able to find a matrix representing a linear transformation by considering the image of the unit square:

  For a transformation represented by a matrix **M**, the image of the point $(1, 0)$ is the first column of **M** and the image of the point $(0, 1)$ is the second column of **M**.
- You should be able to use the matrices representing the following common transformations:

reflection in the line $y = (\tan\theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
horizontal stretch with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch with scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
enlargement with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
anticlockwise rotation of angle θ about the origin ($\theta > 0$)	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
clockwise rotation of angle θ about the origin ($\theta > 0$)	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

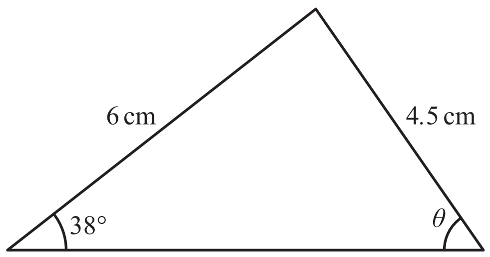
- You should know that the image of a point (x, y) under a translation with vector $\begin{pmatrix} e \\ f \end{pmatrix}$ is $(x+e, y+f)$.
- You should be able to find a matrix representing a composition of two transformations: If the transformation with matrix **A** is followed by the transformation with matrix **B**, the combined transformation has matrix **BA**.
- You should know that $\mathbf{A}^n$ represents transformation **A** repeated n times.
- You should know that, for a transformation represented by matrix **A**, area of image $= |\det \mathbf{A}| \times$ area of object.

Mixed Practice

1 The height of a wave (h m) at a distance (x m) from the shore is modelled by the equation $h = 1.3\sin(2.5x)$.

- a Write down the amplitude of the wave.
- b Find the distance between consecutive peaks of the wave.

2 For the triangle shown in the diagram, find the two possible values of θ.



3 The obtuse angle A has $\sin A = \frac{5}{13}$.

Find the exact value of:

- a $\cos A$
- b $\tan A$

4 Transformation **M** is represented by the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

- a Find the coordinates of the image of the point $(3, -2)$ under the transformation **M**.
- b Find the coordinates of the point whose image is $(2, 1)$.
- c Find the area of the image of the unit square under the transformation $\mathbf{M}^2$.

5
- a Write down the matrix representing the rotation 90° clockwise about the origin.
- b Find the matrix representing transformation **T**, which is the rotation from part **a** followed by an enlargement with scale factor 3.
- c Draw the image of the unit square under transformation **T**.

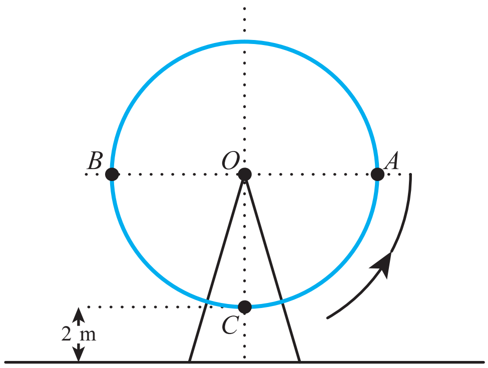
6 Let **R** be the reflection in the line $y = x$ and **S** be the horizontal stretch with scale factor 2.

- a Find the matrix representing the composite transformation when **R** is followed by **S**.
- b A second composite transformation is **S** followed by **R**. Determine whether this is the same transformation as in part **a**.

7 Find the period of the function $f(x) = 3\sin\left(\frac{x}{2}\right) - 4\cos\left(\frac{x}{5}\right)$, where x is in radians.

8 Find the range of the function $f(x) = (4\cos(x - \pi) - 1)^2$.

9 The diagram shows a Ferris wheel that moves with constant speed and completes a rotation every 40 seconds. The wheel has a radius of 12m and its lowest point is 2m above the ground.



Initially, a seat C is vertically below the centre of the wheel, O. It then rotates in an anticlockwise (counterclockwise) direction.

a Write down

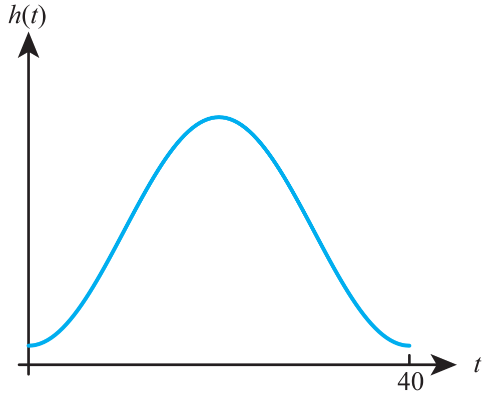
i the height of O above the ground

ii the maximum height above the ground reached by C.

In a revolution, C reaches points A and B, which are at the same height above the ground as the centre of the wheel.

b Write down the number of seconds taken for C to first reach A and then B.

The sketch below shows the graph of function, $\mathrm{h}(t)$, for the height above ground of C, where h is measured in metres and t is the time in seconds, $0 \le t \le 40$.



c **Copy** the sketch and show the results of part **a** and part **b** on your diagram. Label the points clearly with their coordinates.

The height of C above ground can be modelled by the function, $\mathrm{h}(t) = a\cos(bt) + c$, where bt is measured in degrees and t is the time in seconds.

d Find the value of

i a

ii b

iii c.

C **first** reaches a height of 20m above the ground after T seconds.

e **i** Sketch a clearly labelled diagram of the wheel to show the position of C.
ii Find the angle that C has rotated through to reach this position.
iii Find the value of T.

Mathematical Studies SL November 2011 Paper 2 Q5

10 Line l makes a 30º angle with the positive x-axis. Transformation **M** (in two dimensions) is the result of the reflection in the x-axis followed by the reflection in l.

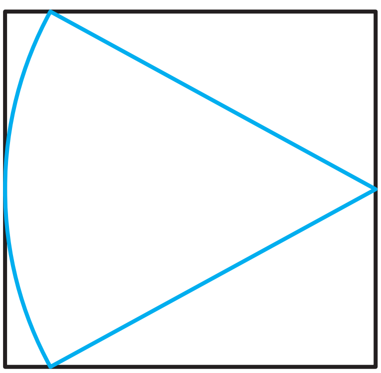
a Find the matrix representing **M**.
b Describe the transformation **M**.
c Describe the transformation resulting from the reflection in l followed by the reflection in the x-axis.

11 Transformation **A** is a 90° rotation clockwise around the origin and transformation **B** is a stretch, scale factor 3, parallel to the y-axis.

a Write down the 2×2 matrices for **A** and **B**.
b Transformation **C** is **A** followed by **B**. Find the matrix for **C**.
c Find the coordinates of the point whose image under **C** is $(6, 12)$.

12 In triangle ABC, $AB = 10\text{cm}$, $AC = 7\text{cm}$ and angle $BAC = 40°$. Find the difference in areas between two possible triangles ABC.

13 A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is 7cm^2, find the dimensions of the rectangle, giving your answers to the nearest millimetre.



Mathematics HL May 2013 Paper 2 TZ1 Q5

14 Prove the identity $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} \equiv \dfrac{2}{\sin^2 x}$.

15 If $0 < k < 1$, find the sum of the solutions to $\sin x = k$ for $-\pi < x < 3\pi$.

16 **a** Sketch $y = x^2 - x$.
b Hence find the values of k for which $\sin^2 x - \sin x - k = 0$ has solutions.

17 Let **S** be the matrix representing reflection in the y-axis and **R** be the matrix representing rotation through 30° anti-clockwise about the origin.

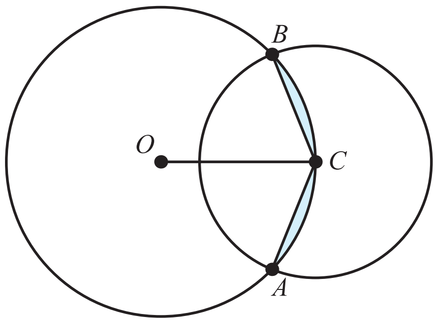
a Find the matrix $\mathbf{T} = \mathbf{R}^{-1}\mathbf{SR}$.

b Describe fully the transformation represented by **T**.

18 Let **M** be the matrix representing reflection in the line $y = x$ and **N** be the matrix representing rotation through 45° anti-clockwise about the origin. Describe fully the transformation repre sented by the matrix $\mathbf{M}^{-1}\mathbf{N}\mathbf{M}$.

19 A bicycle chain is modelled by the arcs of 2 circles connected by 2 straight lines which are tangent to both circles.

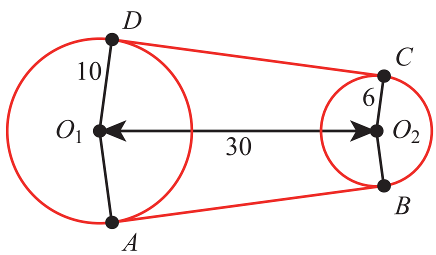
The radius of the larger circle is 10cm and the radius of the smaller circle is 6cm. The distance between the centre of the circles is 30cm.



a Find angle AO_1O_2.

b Hence find the length of the bicycle chain, giving your answer to the nearest cm.

20 The following diagram shows two intersecting circles of radii 4cm and 3cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find:

a $B\hat{O}C$

b the area of the shaded region.

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